BHARATHIDHASANAR MATRIC HIGHER SECONDARY SCHOOL

ARAKKONAM

XII – MATHEMATICS

6 Marks & 10 Marks

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Complex number

EXERCISE: 3.1

Express the following in the standard form a+ib

$$\frac{2(i-3)}{(1+i)^2}$$

Solution: $\frac{2i-6}{1+2i+i^2} = \frac{2i-6}{1+2i-1} = \frac{i-3}{i}$

$= \frac{i-3}{i} \times \frac{-i}{-i} = \frac{-i^2+3i}{-i^2}$

$= 1+3i$

(ii) $\frac{(1+i)(1-2i)}{1+3i}$

Solution : $\frac{(1+i)(1-2i)}{1+3i} = \frac{1-i-2i2}{1+3i} = \frac{3-i}{1+3i} \times \frac{1-3i}{1+3i}$

$= \frac{3-10i-2i^2}{1-9i^2} = \frac{3-10i-3}{1+9}$

$= \frac{-10i}{10} = -i$

(iii) **(-3+i) (4-2i)**

Solution: $(-3+i)(4-2i) = -12+4i+6i-2i^2$

$= -12+10i+2$

$= -10+10i$

(iv) $\dfrac{i^4+i^9+i^{16}}{3-2i^8-i^{10}-i^{15}}$

Solution: $\dfrac{i^4+i^9+i^{16}}{3-2i^8-i^{10}-i^{15}} = \dfrac{1+i+1}{3-2-(i^2)^4-(i^2)^5-(i^2)^7 i}$

$$= \frac{2+i}{1+1+I}$$

$$= \frac{2+i}{2+i} = 1$$

2. find the real and imaginary parts of the following complex number

(i) $\dfrac{1}{1+i}$

Solution: $\dfrac{1}{1+i} = \dfrac{1}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{1-i}{1-i^2}$

$$= \frac{1-i}{1+1}$$

$$= \frac{1}{2}\frac{1}{2} - \frac{i}{2}$$

$$\text{R.P} = \frac{1}{2} \quad \text{I.P} = -\frac{1}{2}$$

(ii) $\dfrac{2+5i}{4-3i}$

solution: $\dfrac{2+5i}{4-3i} \times \dfrac{4+3i}{4-3i} = \dfrac{8+20i+6i+15i^2}{16-9i^2}$

$$= \frac{8+26i-15}{16+9}$$

$$= \frac{-7+26i}{25}$$

$$= \frac{-7}{25} + \frac{26i}{25}$$

R.P = $\frac{-7}{25}$ **; I.P =** $\frac{26}{25}$

(iii)(2+i)(3-2i)

Solution: (2+i) (3-2i) = 6+3i-4i-2i²

$$(2+i)(3-2i) = 6+3i-4i-2i^2$$

$$= 6-i+2$$

$$= 8-i$$

R.P = 8 I.P = -1

3. find the least positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$

Solution: $\left(\frac{1+i}{1-i}\right)^n = 1$

$$\left(\frac{1+i}{1+i}\right)^n = 1$$

$$\left(\frac{1+2i+i^2}{1-i^2}\right)^n = 1$$

$$\left(\frac{1+2i-1}{1+1}\right)^n = 1$$

$$(i)^n = 1$$

Possible value for n = 4,8,12…………….

The least positive integer n is = 4.

Find the real value of x and y for which the following equation are satisfied .

(i) **(1-i) x + (1+i) y = 1-3i**

(1-i) x + (1+i) y = 1-3i

x- ix +y + iy = 1-3i

equating real and imaginary parts

x + y = 1 ……………………………..1

-x + y = -3 ……………………………..2

Solve(1) and(2) 2y = -2

Y = -1 and x = 2

(ii) $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

Solution: $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

$$\frac{(1+i)(3-i)x-2i(3-i)+(2-3i)(3+i)y+i(3+i)}{9-i^2} = i$$

(4+2i)x-6i-2+(9-7i)y+3i-1 = 10i

(4x+9y) + i (2x-7y) = 3+13i

Equating real and imaginary parts

4x + 9y = 3 …………………… (1)

2x- 7y = 13……………………. (2)

Solve 1 and 2 4x + 9y = 3

4x- 14y = 26

(-) (+) (-)

23 y = -23

Y = -1

And 4x-9 = 3 then x = 3

∴ x==3 ; y = 1

(iii) $\sqrt{x^2+3x+8}$ + (x+4) i = y (2+i)

Solution : Equating real and imaginary parts

$\sqrt{x^2+3x+8}$ + (x+4) i = y (2+i)

$\sqrt{x^2+3x+8}$ = 2y…………………………. (1)

x+4 = y ……………………………. (2)

substituting (2) in (1)

$x^2 + 3x + 8 = 4(x+4)^2$

$x^2 + 3x + 8 = 4x^2 + 32x + 64$

$3x^2 + 29x + 56 = 0$

$3x^2 + 21x + 8x + 56 = 0$

$3x(x+7) + 8(x+7) = 0$

$(3x+8)(x+7) = 0$

$3x + 8 = 0$; $x + 7 = 0$

Then $x = -7$ and $y = -\frac{8}{3}$

$x = -7$ sub in equ (1)

$(-7)^2 + 3(-7) + 8 = 4y^2$

$36 = 4y^2$

$Y = 3$

$X = -\frac{8}{3}$ sub in equ (2)

$-\frac{8}{3} + 4 = y$

$\frac{-8+12}{3} = y$ then $x = \frac{4}{3}$

5. for what value of x and y , the number $-3+ix^2y$ and $x^2 + y + 4i$ are complex conjugate of each other .

Solution: $-3+ix^2y$ and $x^2 + y + 4i$ are complex conjugate .

$-3+ix^2y$ = complex conjugate $x^2 + y + 4i$

$-3+ix^2y = x^2 + y - 4i$

Equating real and imaginary parts

$x^2 + y = -3$ ………………………….(1)

$x^2y = -4$

$x^2 = \frac{-4}{y}$ ……………………….(2) sub in equ (1)

$-\frac{4}{y} + y = -3$

$-4 + y^2 = -3y$

$y^2 + 3y - 4 = 0$

$(y+4) + (y-1) = 0$

$Y = 1 , -4$

Sub in equ (2) $x^2 = \frac{-4}{y}$

$Y = 1$ $X^2 = \frac{-4}{1} \therefore X = \pm 2i$

$Y = -4$ $X^2 = \frac{-4}{-4} \therefore x = \pm 1$

Solution $X = \pm 2i$ $Y = 1$ and

$x = \pm 1$ $Y = -4$

EXERCISE 3:2

If $(1+i)(1+2i)(1+3i)(1+4i)..........(1+ni) = x+ iy$. show that $2.5.10..........(1+n^2) = x^2 + y^2$

Solution : given

$(1+i)(1+2i)(1+3i)(1+4i).......... (1+ni) = x+ i y$

Take both side modulus

$|1+i| |1+2i| |1+3i| |1+4i| ....|1+ni| = |x+ i y|$

$(\sqrt{1+1})(\sqrt{1+4})(\sqrt{1+9})(\sqrt{1+16})........(\sqrt{1+n^2}) = \sqrt{x^2+y^2}$

Squaring both sides

(2)(5)(10)(16)...... $(1+n^2)$ = (x^2+y^2)

2. find the square root of (-8-6i).

Solution: let $\sqrt{(-8-6i)} = x + iy$

$$-8-6i = (x+iy)^2$$

$$= x^2 - y^2 + 2ixy$$

Equating real and imaginary parts

$$x^2 - y^2 = -8$$

$$2xy = -6$$

$$Y = -\frac{3}{x}$$

Put $t = x^2$ .... (a)

$$t - \frac{9}{t} = -8$$

$$t^2 - 9 = -8t$$

$$t^2 + 8t - 9 = 0$$

$$(t+9)(t-1) = 0$$

$$t = -9, 1$$

put $t = -9$ in(a) $t = x^2$	**put** $t = 1$
$x^2 = -9$	$x^2 = 1$
$x = \pm 3i$	$x = \pm 1$

x = 3i => y = i

x =- 3i => y = -i

x = 1 => y = -3

x =- 1 => y = 3

the square root of -8-6i = (1-3i) and

-8-6i = (-1+3i)

3. if z^2 = (0,1) find z .

Solution : let z = (a, b)

$Z^2 = (a, b)\,(a, b)$

$Z2 = (a^2 - b^2, ab + ab)$

$= (a^2 - b^2, 2ab)$

If $z^2 = (0,1) \Rightarrow a^2 - b^2$ and $2ab = 1$

$a^2 = b^2 \Rightarrow ab = \frac{1}{2}$

$a = b \Rightarrow a \cdot a = \frac{1}{2}$

$\Rightarrow a^2 = \frac{1}{2}$

$Z = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $Z = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

$a = \pm\frac{1}{\sqrt{2}}$

hence Z = $(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$; $(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$

4. prove that triangle formed by the point representing the complex number (10+8i)

(-2+4i)and(-11+31i) on the argand plane right angled.

Solution :

Let A,B,C representing the complex number (10+8i) (-2+4i)and(-11+31i) on the argand diagram .

AB = (10+8i) – (-2+4i)

= 12+4i

= $\sqrt{144+16}$ = $\sqrt{160}$

BC = (-2+4i) – (-11+31i)

= 9-27i

= $\sqrt{81+729}$ = $\sqrt{810}$

CA = (-11+31i) – (10+8i)

= -21+23i

= $\sqrt{441+529}$ = $\sqrt{970}$

$AB^2 + BC^2 = CA^2$

= >Δ ABC is right angled .

5. prove that the point representing the complex number (7+5i)(5+2i)and(4+7i) and (2+4i) form a Parallelogram. (plot the point and use midpoint formula).

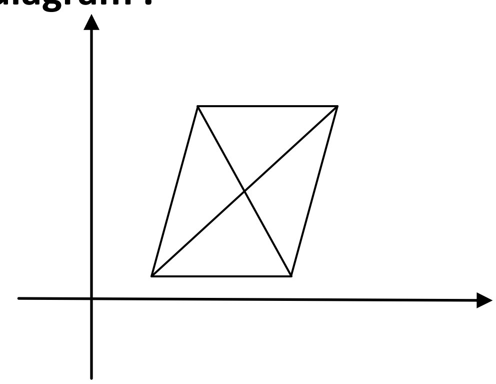
Solution: let A,B,C,D represent the complex number (7+5i)(5+2i)and(4+7i) and (2+4i) in a argand diagram .

(7+5i) => (7,5)

(5+2i) => (5,2)

(4+7i) => (4,7)

(2+4i) => (2,4)



Midpoint of AD is

$$\left(\frac{2+7}{2}, \frac{4+5}{2}\right) = \left(\frac{9}{2}, \frac{9}{2}\right)$$

Midpoint of BC is

$$\left(\frac{5+4}{2}, \frac{2+7}{2}\right) = \left(\frac{9}{2}, \frac{9}{2}\right)$$

since the midpoint of the diagonal are same . then the diagonal are bisect each other .

∴ABCD is a parallelogram .

Express the following complex number in polar form .

(i) $2+2\sqrt{3}$ i , (ii) -1+i$\sqrt{3}$, ,(iii) 1-i (iv) -1-i

SOLUTION: $|r| = z$

$= |2+2\sqrt{3}\,i|$

$= \sqrt{4+12} = 4$

$\alpha = \tan^{-1}\frac{b}{a}$

$=> \alpha = \tan^{-1}\frac{2\sqrt{3}}{2}$

$=> \alpha = \tan^{-1}\sqrt{3}$ $\quad => \alpha = \frac{\pi}{3}$

$\theta = \alpha = \frac{\pi}{3}$

polar form: $z = r[\cos\theta + \sin\theta] => 2+2\sqrt{3}\,i = 4\left[\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right]$

(ii) SOLUTION: $r = |z|$

$= |-1 + i\sqrt{3}|$

$= \sqrt{1+3} = 2$

$\alpha = \tan^{-1}\frac{b}{a}$

$=> \alpha = \tan^{-1}\frac{\sqrt{3}}{1}$

$=> \alpha = \tan^{-1}\sqrt{3}$ $\quad => \theta = \pi - \alpha$

$=> \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

polar form : $z = r[\cos\theta + \sin\theta] => -1+\sqrt{3}\,i = 2\left[\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right]$

(iii) SOLUTION: $|r| = z$

$= |1 + i|$

$= \sqrt{1+1} = \sqrt{2}$

$\alpha = \tan^{-1}\frac{b}{a}$

$=>\alpha = \tan^{-1}\frac{1}{1}$

$\theta = -\alpha$ $=>\alpha = -\frac{\pi}{3}$

polar form : z = $r[\cos\theta + \sin\theta]$=> 1+ i = $\sqrt{2}\left[\cos\frac{-\pi}{3} + \sin\frac{-\pi}{3}\right]$

(iv) SOLUTION: $|r| = z$

$= |-1 -i|$

$= \sqrt{1+1} = \sqrt{2}$

$\alpha = \tan^{-1}\frac{b}{a}$

$=>\alpha = \tan^{-1}\frac{-1}{-1}$

$=>\alpha = -\frac{\pi}{3}$ $=>\theta = -\alpha + \pi$

$=>\theta = \frac{-3\pi}{4}$

polar form : z = $r[\cos\theta + \sin\theta]$ => -1- i = $\sqrt{2}\left[\cos\frac{-3\pi}{4} + \sin\frac{-3\pi}{4}\right]$

7. if arg (z-1) = $\frac{\pi}{6}$ and arg (z+1) = $\frac{2\pi}{3}$ then prove that $|z| = 1$.

Solution: arg (z-1) = $\frac{\pi}{6}$

arg (x+iy-1) = $\frac{\pi}{6}$

$=>\tan^{-1}\frac{y}{x-1} = \frac{\pi}{6}$

$=> \frac{y}{x-1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ ……………………….(1)

=> x-1 = $y\sqrt{3}$

arg (z+1) = $\frac{2\pi}{3}$

arg (x+iy+1) = $\frac{2\pi}{3}$

$=>\tan^{-1}\left(\frac{y}{x+1}\right) = \frac{2\pi}{3}$

$=> \frac{y}{x+1} = \tan\frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right)$

$=> \frac{y}{x+1} = -\tan\frac{\pi}{3} = -\sqrt{3}$

$\sqrt{3}$(x+1) = -y ……………………. (2)

Multiply (1) and (2)

(x-1) $\sqrt{3}$(x+1) = - $y\sqrt{3}$ y

$X^2 -1 = -y^2 => x^2 + y^2 = 1$

$|z|^2 = 1 => |z| = 1$

8.P represent the variable complex number z. find the locus of P , if

(i) $\text{Im}\left(\frac{2z+1}{iz+1}\right) = -2$ (ii) $|z - 5i| = |z + 5i|$ (iii) $\text{Re}\left(\frac{z-1}{z+i}\right) = 1$

(iv) $|2z-3| = 2$, (v) are $\left(\frac{z-1}{z+3}\right) = \frac{\pi}{2}$

Solution :

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = -2$$

Let z = |x+iy|

$$\text{Im}\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = -2$$

$$\text{Im}\left(\frac{2x+2iy+1}{ix-y+1}\right) = -2$$

$$\text{Im}\left(\frac{2x+2iy+1}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right) = -2$$

$$\left(\frac{(2x+1)(1-y)+2xy+i\{2y(1-y)-x(2x+1)\}}{(1-y)^2+x^2}\right) = -2$$

$2y-2y^2-2x^2-x = -2(1-2y+y^2+x^2)$

$2y-2y^2-2x^2-x = -2+4y-2y^2-2x^2$

$-x-2y = -2 \Rightarrow x+2y = 2$

(ii) let z = x + iy $|z-5i| = |z+5i|$

$|X+iy-5i| = |x+iy+5i|$

$|X+i(y-5)| = |x+(y+5)|$

$$\sqrt{x^2+(y-5)^2} = \sqrt{x^2+(y+5)^2}$$

Squaring both sides

$X^2+y^2-10y+25 = X^2+y^2+10y+25$

$-20y = 0$

$\Rightarrow y = 0$

(iii) $\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = -2$

Let z = x+iy

Re $\left(\frac{(x+iy)+1}{(x+iy)+i}\right) = 1$

Re$\left(\frac{x+1+iy}{x+i(y+1)}\right) = 1$

Re$\left(\frac{(x+1)+iy}{x+i(y+1)} \times \frac{x-i(y+1)}{x-i(y+1)}\right) = 1$

Re $\left(\frac{x(x+1)+y(y+1)+i(xy-(x+1)(y+1))}{x^2+(y+1)^2}\right) = 1$

$\frac{x(x+1)+y(y+1)}{x^2+(y+1)^2} = 1$

$X^2 + x + y^2 + y = x^2 + y^2 + 2y + 1$

x-y = 1

(iv) let z = x + i y

2z - 3 = 2 => 2 (x+ i y) -3 = 2

2x-3 + 2iy = 2

$\sqrt{(2x-3)^2+(2y)^2} = 2$

Squaring on the both sides

$$(2x-3)^2 + (2y)^2 = 4$$

$$4x^2-12x+9+4y^2 = 4$$

$$4x^2+4y^2-12x+5 = 0$$

(v)

$$\text{are}\left(\frac{z-1}{z+3}\right) = \frac{\pi}{2}$$

$$z = x+iy$$

$$\text{are}\left(\frac{x+iy-1}{x+iy+3}\right) = \frac{\pi}{2}$$

$$\text{are}\left(\frac{(x-1)+iy}{x+3+iy}\right) = \frac{\pi}{2}$$

$$\text{are}\left(\frac{((x-1)+iy)((x+3)-iy)}{(x+3+iy)((x+3)-iy))}\right) = \frac{\pi}{2}$$

$$\text{are}\left(\frac{((x-1)(x+3)+y^2+iy(x+3)-y(x-1))}{(x+3)^2+y^2}\right) = \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{y(x+3)-yy(x-1)}{(x-1)(x+3)+y^2}\right) = \frac{\pi}{2}$$

$\left(\frac{y(x+3)-yy(x-1)}{(x-1)(x+3)+y^2}\right)$ = $\tan\frac{\pi}{2}$

$\frac{y(x+3)-y(x-1)}{(x-1)(x+3)+y^2}$ = ∞ (since $\tan^{-1}\infty = \frac{\pi}{2}$)

$\frac{(x-1)(x+3)+y^2}{y(x+3)-y(x-1)}$ = $\frac{1}{0}$

(x-1)(x+3) + y^2 = 0 => $x^2 + y^2 + 2x - 3 = 0$

Prove that the complex numbers 3+3i , -3-3i , -3$\sqrt{3}$ + 3$\sqrt{3}$i are the vertices of an equilateral triangle in the complex plane .

Solution : let A , B ,C represent the complex numbers 3+3i , -3-3i , -3$\sqrt{3}$ + 3$\sqrt{3}$i in the Argand diagram .

AB = |(3+3i) – (-3-3i)|

= 6+6i = $\sqrt{72}$

BC = | (-3-3i) -(-3$\sqrt{3}$ + 3$\sqrt{3}$i) |

= $\sqrt{72}$

CA = | (-3$\sqrt{3}$ + 3$\sqrt{3}$i) –(3+3i) |

= $\sqrt{72}$

AB = BC = CA

∴ Δ ABC is an equilateral triangle .

Prove that the complex numbers 2i , 1+i , 4+4i and 3+5i are the vertices of an rectangle in the Argand plane .

Solution : let A , B ,C represent the complex numbers 2i , 1+i , 4+4i and 3+5i in the Argand diagram.

$AB = |2i - (1+i)|$

$= |-1 + i| = \sqrt{2}$

$BC = |(1+i) - (4+4i)|$

$= |-3-3i| = \sqrt{18}$

$CD = |(4+4i) - (3+5i)|$

$= |1 - i| = \sqrt{2}$

$DA = |(3+5i) - (2i)|$

$= |3+3i| = \sqrt{18}$

AB = BC= CD = DA

$AC = |(0+2i) - (4+4i)|$

$= |-4-2i| = \sqrt{20}$

$AB^2 + BC^2 = 2 + 18 = 20 = AC^2$

Hence $AB^2 + BC^2 = AC^2$

As pairs of opposite sides are equal

ABCD is a rectangle .

Show that the complex numbers 7+9i , -3+7i ,3+3i form a right angledtriangle on the Argand diagram .

Solution :

let A , B ,C represent the complex numbers 7+9i , -3+7i ,3+3i in the Argand diagram .

AB =|(7+9i)–(-3+7i)

= 10+2i| = $\sqrt{10^2 + 2^2}$ **=** $\sqrt{104}$

BC =|(-3+7i) – (3+3i)|

=|-6 +4i |= $\sqrt{(-6)^2 + 4^2}$ **=** $\sqrt{52}$

CA =|(3+3i) – (7+9i)| = $\sqrt{(-6)^2 + (-4)^2}$ **=** $\sqrt{52}$

$(AB)^2 = (BC)^2 + (CA)^2$

$\angle$**BCA** = $(90)^\circ$

Hence ∴ Δ ABC is a right angled isosceles triangle .

Find the square root of -7 + 24i (-7 + 24i)

Solution :

$\sqrt{-7 + 24i}$ **= x+iy**

Squaring on both sides

-7 + 24i = $(x^2 - y^2)$ **+ 2ixy**

$(x^2 - y^2)$ **= -7 and 2xy = 24**

$(x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$

$= \sqrt{(-7)^2 + (24)^2} = 25$

Solve $(x^2 - y^2) = -7$ **and** $(x^2 + y^2) = 25$

We get $x = \pm 3$, $y = \pm 4$

Since xy is positive , x ,y have the same sign.

$(x = 3 , y = 4)$ **or** $(x = -3 , y = -4)$

$\sqrt{-7 + 24i} = 3+4i$ **or** $-3-4i$

$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+ iB$ **, prove that**

(i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots\ldots\ldots(a_n^2 + b_n^2) = A^2+B^2$

(ii) $\tan^{-1}(\frac{b_1}{a_1}) + \tan^{-1}(\frac{b_2}{a_2}) \ldots\ldots\ldots + \tan^{-1}(\frac{b_n}{a_n}) = k\pi + \tan^{-1}(\frac{B}{A})$, $k \in R$

Solution :

$$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+ iB$$

$$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+ iB$$

$$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+ iB$$

$$\sqrt{(a_1^2 + b_1^2)} + \sqrt{(a_2^2 + b_2^2)} \ldots\ldots\ldots \sqrt{(a_n^2 + b_n^2)}$$

Squaring on both sides

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots\ldots\ldots(a_n^2 + b_n^2) = A^2+B^2$$

Also

$\arg[(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n)] = \arg(A+iB)$

$\arg(a_1+ib_1)+\arg(a_2+ib_2)\ldots\ldots\ldots+ \arg(a_n+ib_n) = \arg(A+iB)$

$\arg(a_i+ib_i) = \tan^{-1}(\frac{b_i}{a_i})$

$\tan^{-1}(\frac{b_1}{a_1}) + \tan^{-1}(\frac{b_2}{a_2}) \ldots\ldots\ldots + \tan^{-1}(\frac{b_n}{a_n})$

SIX MARKS

Prove that the complex numbers 3+3i , -3-3i , $-3\sqrt{3} + 3\sqrt{3}i$ are the vertices of an equilateral triangle in the complex plane .

Solution : let A , B ,C represent the complex numbers 3+3i , -3-3i , $-3\sqrt{3} + 3\sqrt{3}i$ in the Argand diagram .

AB = (3+3i) – (-3-3i)

$= 6+6i = \sqrt{72}$

BC = (-3-3i) $-(-3\sqrt{3} + 3\sqrt{3}i)$

$= \sqrt{72}$

CA = $(-3\sqrt{3} + 3\sqrt{3}i) - (3+3i)$

$= \sqrt{72}$

AB = BC = CA

∴ Δ ABC is an equilateral triangle .

Prove that the complex numbers 2i , 1+i , 4+4i and 3+5i are the vertices of an rectangle in the Argand plane .

Solution : let A , B ,C represent the complex numbers 2i , 1+i , 4+4i and 3+5i in the Argand diagram .

AB = 2i – (1+i)

= -1 +i = $\sqrt{2}$

BC = (1+i)- (4+4i)

= -3-3i = $\sqrt{18}$

CD = (4+4i) – (3+5i)

= 1- i = $\sqrt{2}$

DA = (3+5i) –(2i)

= 3+3i = $\sqrt{18}$

AB = BC= CD = DA

AC = (0+2i) –(4+4i)

= -4-2i = $\sqrt{20}$

$AB^2 + BC^2$ **= 2 +18 = 20 =**AC^2

Hence $AB^2 + BC^2 = AC^2$

As pairs of opposite sides are equal

ABCD is a rectangle .

Show that the complex numbers 7+9i , -3+7i ,3+3i form a right angledtriangle on the Argand diagram .

Solution :

let A , B ,C represent the complex numbers 7+9i , -3+7i ,3+3i in the Argand diagram .

AB = (7+9i)–(-3+7i)

= 10+2i = $\sqrt{10^2 + 2^2}$ = $\sqrt{104}$

BC = (-3+7i) – (3+3i)

= -6 +4i = $\sqrt{(-6)^2 + 4^2}$ = $\sqrt{52}$

CA = (3+3i) – (7+9i) = $\sqrt{(-6)^2 + (-4)^2}$ = $\sqrt{52}$

$(AB)^2 = (BC)^2 + (CA)^2$

$\angle$**BCA** = $(90)^\circ$

Hence ∴ Δ ABC is a right angled isosceles triangle .

Find the square root of -7 + 24i (-7 + 24i)

Solution :

$\sqrt{-7 + 24i}$ **= x+iy**

Squaring on both sides

-7 + 24i = $(x^2 - y^2)$ **+ 2ixy**

$(x^2 - y^2) = -7$ and $2xy = 24$

$(x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$

$= \sqrt{(-7)^2 + (24)^2} = 25$

Solve $(x^2 - y^2) = -7$ and $(x^2 + y^2) = 25$

We get $x = \pm 3$, $y = \pm 4$

Since xy is positive , x ,y have the same sign.

$(x = 3 , y = 4)$ or $(x = -3 , y = -4)$

$\sqrt{-7+24i} = 3+4i$ or $-3-4i$

20. $(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+iB$, prove that

(i) $(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots\ldots\ldots(a_n^2+b_n^2) = A^2+B^2$

(ii) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) \ldots\ldots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = k\pi + \tan^{-1}\left(\frac{B}{A}\right)$, $k \in R$

Solution :

$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+iB$

$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+iB$

$(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n) = A+iB$

$\sqrt{(a_1^2+b_1^2)} + \sqrt{(a_2^2+b_2^2)} \ldots\ldots\ldots \sqrt{(a_n^2+b_n^2)}$

Squaring on both sides

$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots\ldots\ldots(a_n^2+b_n^2) = A^2+B^2$

Also

$\arg[(a_1+ib_1)(a_2+ib_2)\ldots\ldots\ldots(a_n+ib_n)] = \arg(A+iB)$

$\arg(a_1+ib_1)+\arg(a_2+ib_2)\ldots\ldots\ldots+\arg(a_n+ib_n) = \arg(A+iB)$

$\arg(a_i+ib_i) = \tan^{-1}\left(\frac{b_i}{a_i}\right)$

$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) \ldots\ldots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right)$

EXERCISE 3:3

1. **Solve the equation $x^4 - 8x^3 + 24x^2 - 32x + 20 = 0$ if $3+i$ is a root**

Solution:

Since $3+i$ is a root and

$3-i$ is also a root

Sum of the roots $= 3+i+3-i = 6$

Product of the roots $= (3+i)(3-i)\ 9+1 = 10$

The corresponding factor is $x^2 - 6x + 10$

The other factor is $x^2 + px + 2$

Hence $x^4 - 8x^3 + 24x^2 - 32x + 20 = (x^2 - 6x + 10)(x^2 + px + 2)$

Equating the coefficient of x

$-32 = 10p - 12 \Rightarrow p = -2$

$\therefore$ The other factor is $x^2 - 2x + 2$

Solve x = $\frac{2\pm\sqrt{4-8}}{2}$ = $\frac{2\pm 2i}{2}$

= $1\pm i$

The roofs are $3\pm$ I and $1\pm i$.

2. **Solve the equation $x^4 -4x^3 +11x^2 -14x +10 = 0$ if 1+2i is a root**

Solution:

Since 1+2i is a root and

1-2i is also a root

Sum of the roots = 1+2i+1-2i = 2

Product of the roots = (1+2i)(1-2i) =1+4 =5

The corresponding factor is $x^2 -4x + 5$

The other factor is $x^2 +p x +2$

Hence $x^4 -8x^3 +24x^2 -32x +20 = (x^2 -2x + 5)(x^2 +px +2)$

Equating the coefficient of x

-14 = -4+5p => p = -2

∴ The other factor is $x^2 -2x +2$

Solve x = $\frac{2\pm\sqrt{4-8}}{2}$ = $\frac{2\pm 2i}{2}$ = $1\pm i$The roofs are $1\pm$2i and $1\pm i$.

3. Solve the equation $6x^4 - 25x^3 + 32x^2 + 3x - 10 = 0$ if 2-i is a root

Solution:

Since 2-i is a root and

2+i is also a root

Sum of the roots = 2-i+2+i= 4

Product of the roots = (2-i)(2+i) =1+4 =5

The corresponding factor is $x^2 - 4x + 5$

The other factor is $6x^2 + px - 2$

Hence $x^4 - 8x^3 + 24x^2 - 32x + 20 = (x^2 - 4x + 5)(6x^2 + px - 2)$

Equating the coefficient of x

3 = 5p+ 8 => p = -1

∴ The other factor is $6x^2 - x - 2$

Solve $x = \frac{1 \pm \sqrt{1+48}}{12} = \frac{1 \pm 7}{12}$

$= \frac{8}{12}$ **or** $-\frac{6}{12}$

∴ The roofs are $2 \pm i$ and $-\frac{1}{2}, \frac{2}{2}$.

EXERCISE : 3.4

1.Simplify : $\frac{(\cos 2\theta - isin2\theta)^7 (cos3\theta + isin3\theta)^{-5}}{(cos4\theta + ision4\theta)^{12} (cos5\theta -- isin5\theta)^{-6}}$

Solution : $\frac{(\cos 2\theta - isin2\theta)^7 (cos3\theta+isin3\theta)^{-5}}{(cos4\theta+ision4\theta)^{12}(cos5\theta--isin5\theta)^{-6}} = \frac{(\cos\theta isin\theta)^{-14}(cos\theta+isin\theta)^{-15}}{(cos\theta+ision\theta)^{48}(cos\theta+isin\theta)^{30}}$

= $(\cos\theta isin\theta)^{-107}$

= $\cos(-107)\theta + isin(-107)\theta$

2. simplify : $\frac{(\cos\alpha+i\sin\alpha)^3}{(\sin\beta+i\cos\beta)^4}$

Solution : $\frac{(\cos\alpha+i\sin\alpha)^3}{(\sin\beta+i\cos\beta)^4} = \frac{(\cos\alpha+i\sin\alpha)^3}{(\cos\left(\frac{\pi}{2}-\beta\right)+i\sin\left(\frac{\pi}{2}-\beta\right))^4})$

$= \frac{(COS3\alpha+i\sin3\alpha))}{(cos4\left(\frac{\pi}{2}-\beta\right)+i\sin4\left(\frac{\pi}{2}-\beta\right))}$

= $(\cos(3\alpha - 2\pi + 4\beta) + i\sin(3\alpha - 2\pi + 4\beta))$

= $(\cos(3\alpha + 4\beta) + i\sin(3\alpha + 4\beta))$

(3) If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(iv) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(Hints : Take $a = \text{cis}\,\alpha$, $b = \text{cis}\,\beta$, $c = \text{cis}\,\gamma$

$$a + b + c = 0 \Rightarrow a^3 + b^3 + b^3 = 3abc$$

$$1/a + 1/b + 1/c = 0 \Rightarrow a^2 + b^2 + c^2 = 0)$$

(v) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

Solution: given $\cos\alpha + cos\beta + cos\gamma = 0$

$\sin\alpha + sin\beta + sin\gamma = 0$

$\therefore$ $(\cos\alpha + cos\beta + cos\gamma) + i(\sin\alpha + sin\beta + sin\gamma) = 0$

$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$

Let $a = (\cos\alpha + i\sin\alpha)$; $b = (\cos\beta + i\sin\beta)$; $c = (\cos\gamma + i\sin\gamma)$

Then $a + b + c = 0$

But $a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc+ca)$

$= 0$ (since $a+b+c = 0$)

$\therefore a^3+b^3+c^3 = 3abc$

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3\ (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\text{Cos}3\alpha + i\sin3\alpha + \text{Cos}3\beta + i\sin3\beta + \text{Cos}3\gamma + i\sin3\gamma$

$= 3(\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$

Equating real and imaginary parts ,we have

$\text{Cos}3\alpha + \text{Cos}3\beta + \text{Cos}3\gamma = 3\cos(\alpha+\beta+\gamma)$

$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha+\beta+\gamma)$

(iii) (iv) $a = \cos\alpha + i\sin\alpha \Rightarrow \frac{1}{a} = \cos\alpha - i\sin\alpha$

$b = \cos\beta + i\sin\beta \Rightarrow \frac{1}{b} = \cos\beta - i\sin\beta$

$c = \cos\gamma + i\sin\gamma \Rightarrow \frac{1}{c} = \cos\gamma - i\sin\gamma$

Hence $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = (\text{Cos}\alpha + \text{Cos}\beta + \text{Cos}\gamma) - i\ (\sin\alpha + \sin\beta + \sin\gamma)$

But $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2$

$$a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$$

$$a^2 + b^2 + c^2 + 2abc\,(0) = 0$$

$$a^2 + b^2 + c^2 = 0$$

$$(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$

$$\text{Cos}2\alpha + i\sin 2\alpha + \text{Cos}\beta + i\sin 2\beta + \text{Cos}2\gamma + i\sin 2\gamma = 0$$

$$\text{Cos}2\alpha + \text{Cos}2\beta + \text{Cos}2\gamma + i\,(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$

Equating real and imaginary parts ,we have

$$\text{Cos}2\alpha + \text{Cos}2\beta + \text{Cos}2\gamma = 0$$

(V) from equation (1) we get

$$(2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\beta - 1) = 0$$

$$2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 3$$

$$2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$$

$$(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = \frac{3}{2}$$

Further $1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = \frac{3}{2}$

$$3 - \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$

$$\frac{6-3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$$

prove that

(1) $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cdot \cos\left(\frac{n\pi}{4}\right)$

(2) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^n \cdot \cos\left(\frac{n\pi}{3}\right)$

(3) $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n-1} \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{n\theta}{2}\right)$.

(4) $(1+i)^{4n}$ and $(1+i)^{4n+2}$ are real and purely imaginary respectively

Solution : (1) $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cdot \cos\left(\frac{n\pi}{4}\right)$

: $r = |z|$

$= |1+i|$

$= \sqrt{1+1} = \sqrt{2}$

$\alpha = \tan^{-1}\frac{b}{a}$

$\Rightarrow \alpha = \tan^{-1}\frac{1}{1}$

$\theta = \alpha \quad \Rightarrow \alpha = \frac{\pi}{4}$

polar form :

$z = r[\cos\theta + \sin\theta] \Rightarrow (1+i)^n = \sqrt{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n \ldots\ldots\ldots\ldots\ldots(1)$

lll^{ly} $(1-i)^n = \sqrt{2^n}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n \ldots\ldots\ldots\ldots\ldots(2)$

(1)+(2) $(1+i)^n + (1-i)^n = \sqrt{2^n}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)^n + \sqrt{2^n}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$

$= \sqrt{2^n}\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$

$= \sqrt{2^n}\, 2\cos\frac{n\pi}{4}$

$= \quad 2^{\frac{n}{2}}\, 2\cos\frac{n\pi}{4}$

$= 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}$

(2) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^n . \cos\left(\frac{n\pi}{3}\right)$

Solution :

: $|r| = z$

$= |1 + i\sqrt{3}|$

$= \sqrt{1+3} = \sqrt{4} = 2$

$\alpha = \tan^{-1}\frac{b}{a}$

$\Rightarrow \alpha = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3}$

$\alpha = \frac{\pi}{3} \qquad \Rightarrow \theta = \alpha$

polar form :

$z = r[\cos\theta + \sin\theta] \Rightarrow (1+ i\sqrt{3})^n = 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n \ldots\ldots\ldots\ldots$ **(1)**

ll^{ly} $(1- i\sqrt{3})^n = 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n$.......................**(2)**

$(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^n(\cos\frac{\pi}{3} + \sin\frac{\pi}{3})^n + 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n$

$= \quad (\cos\frac{n\pi}{3} + \sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3})$

$= 2^n 2\cos\frac{n\pi}{3}$

$= \quad 2^n 2\cos\frac{n\pi}{3}$

$= 2^{n+1}\cos\frac{n\pi}{3}$

(3) $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n-1} \cos(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$.

Solution : L. H .S

$(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$

Since $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ **and** $\sin\theta = 2 \cos(\frac{\theta}{2}) \cos(\frac{\theta}{2})$

$= (2\cos^2\frac{\theta}{2} + i\, 2 \cos(\frac{\theta}{2}) \cos(\frac{\theta}{2}))^n + (2\cos^2\frac{\theta}{2} - i\, 2 \cos(\frac{\theta}{2}) \cos(\frac{\theta}{2}))^n$

$= 2^n\cos^n\frac{\theta}{2} \left((\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})^n + (\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})^n \right)$

$= 2^n\cos^n\frac{\theta}{2} \left((\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}) + (\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}) \right)$

$= \quad 2^n\cos^n\frac{\theta}{2} \; (2\cos\frac{n\theta}{2})$

$= \quad 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$

(iv) $(1+ i)^{4n}$ **real part**

Z = 1+i

: r = | z |

$= |1 + i|$

$= \sqrt{1+1} \quad = \sqrt{2}$

$\alpha = \tan^{-1}\frac{b}{a}$

$=> \alpha = \tan^{-1}\frac{1}{1}$

$\theta = \alpha \quad => \alpha = \frac{\pi}{4}$

polar form : $z = r[\cos\theta + \sin\theta]$

$=> (1+i)^{4n} = (\sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}))^{4n}$

$= 2^{2n}((\cos\frac{4n\pi}{4} + i\sin\frac{4n\pi}{4})$

$= 2^{2n}((\cos n\pi + i\sin n\pi)$

$= 2^{2n}\cos n\pi \qquad (\sin n\pi = 0)$

= purely real

$(1+i)^{4n+2} = (\sqrt{2}\,(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}))^{4n+2}$

$= 2^{2n+1}(\cos(n\pi + \frac{\pi}{2}) + i\sin(n\pi + \frac{\pi}{2}))$

$= 2^{2n+1}((i\cos n\pi + \sin n\pi)$

$= i2^{2n}\cos n\pi$

= **purely imaginary .**

(5) if $\alpha and \beta$ are the root of the equation $x^2 - 2px + (p^2 + q^2) = 0$ and $\tan\theta = \frac{q}{y+p}$

Show that $\frac{((y+\alpha)^n)-((y+\beta)^n)}{\alpha-\beta}$ = $q^{n-1}\frac{\sin n\theta}{\sin^n\theta}$

Solution : $x^2 - 2px + (p^2 + q^2) = 0$

Let $\alpha and \beta$ be the roots of the equation

Then $\alpha+\beta$ = 2p

$\alpha\beta$ = p2 + q2

a = 1 ; b = 2p ; c = $p^2 + q^2$

$$= \frac{2p\pm\sqrt{4p^2-4p^2+4q^2}}{2}$$

$$= \frac{2p\pm\sqrt{4q^2)}}{2}$$

$= p\pm iq$

$$\alpha = p + iq and \beta = p - iq$$

and $\alpha-\beta$ = 2 iq

Given $\tan\theta$ = $\frac{q}{y+p}$

y + p = $\frac{q}{\tan\theta}$ => y + p = q $\frac{\sin\theta}{\cos\theta}$

$$\frac{((y+\alpha)^n)-((y+\beta)^n)}{\alpha-\beta} = \frac{(y+p+iq)^n-(y+p-iq)^n}{2iq}$$

$$= \frac{(q\frac{\sin\theta}{\cos\theta}+iq)^n-(q\frac{\sin\theta}{\cos\theta}-iq)^n}{2iq}$$

$$= \frac{\left(\frac{q\cos\theta+iq\sin\theta}{\sin\theta})^n-(\frac{q\cos\theta-iq\sin\theta}{\sin\theta})^n\right)}{2iq}$$

$$= \frac{q^n((cos\theta+isin\theta)^n-(cos\theta-isin\theta)^n)}{2iq\sin^n\theta}$$

$$= \frac{q^{n-1}}{2i\sin^n\theta}(\cos n\theta + i\sin n\theta - \cos n\theta + i\cos n\theta)$$

$$= \frac{q^{n-1}}{2i\sin^n\theta}(2i\sin n\theta)$$

$$= q^{n-1}\frac{\sin n\theta}{\sin^n\theta}$$

(6) If $\alpha and \beta$ are the roots $x^2 - 2x + 4 = 0$. prove that $\alpha^n - \beta^n$ = I $2^{n+1}\sin\frac{n\pi}{3}$ and deduct $\alpha^9 - \beta^9$.

Solution : $x^2 - 2x + 4 = 0.$

a = 1 ; b = -2 c = 4

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2\pm\sqrt{4-16}}{2}$$

$$x = \frac{2\pm\sqrt{-12}}{2}$$

$$x = \frac{2\pm i2\sqrt{3}}{2}$$

$$x = 1\pm i\sqrt{3}$$

$$x = 1+i\sqrt{3} \quad ; \quad x = 1-i\sqrt{3}$$

r = | z|

$= \left| 1 + i\sqrt{3} \right|$

$= \sqrt{1+3} \quad = \sqrt{4} = 2$

$m = \tan^{-1}\frac{b}{a}$

$\Rightarrow m = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3}$

$m = \frac{\pi}{3} \quad \Rightarrow \theta = m = \frac{\pi}{3}$

polar form :

$z = r[\cos\theta + \sin\theta] \quad \Rightarrow (\alpha)^n = 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n \ldots\ldots\ldots\ldots\ldots (1)$

$lll^{ly}(\beta)^n = 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

$(1)+(2)(\alpha)^n - (\beta)^n = 2^n(\cos\frac{\pi}{3} + \sin\frac{\pi}{3})^n - 2^n(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^n$

$= (\cos\frac{n\pi}{3} + \sin\frac{n\pi}{3} - \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3})$

$= i\, 2^n(2\sin\frac{n\pi}{3})$

$= i2^{n+1}\sin\frac{n\pi}{3}$

$(\alpha)^9 - (\beta)^9 = i2^{9+1}\sin\frac{n\pi}{3}$

$= i2^{10}\sin\frac{9\pi}{3}$

$= i2^{10}\sin(3\pi) = 0$

(7) If $x + \frac{1}{x} = 2\cos\theta$ prove that

(i) $x^n + \frac{1}{x^n} = 2\cos n\theta$ (ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$

Solution : put x = $\cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$X + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$x^n = (\cos\theta + i\sin\theta)^n$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

(i) $x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) $x^n - \frac{1}{x^n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$

(8) If $x + \frac{1}{x} = 2\cos\theta$ and $y + \frac{1}{y} = 2\cos\phi$ show that

(i) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$ (ii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\theta - n\phi)$

Solution: put $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$X+\frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

put $y = \cos\varphi + i\sin\varphi$

$$\frac{1}{y} = \cos\varphi - i\sin\varphi$$

$$y+\frac{1}{y} = \cos\varphi + i\sin\varphi + \cos\varphi - i\sin\varphi$$

$$= 2\cos\varphi$$

$$\frac{x^m}{y^n} = \frac{(\cos\theta+i\sin\theta)^m}{(\cos\varphi-i\sin\varphi)^n}$$

$$= \frac{(\cos m\theta+i\sin m\theta)}{(\cos n\varphi-i\sin n\varphi)}$$

$$= (\cos(m\theta - n\varphi) + i\sin(m\theta - n\varphi))$$

$$\frac{y^n}{x^m} = (\cos(m\theta - n\varphi) - i\sin(m\theta - n\varphi))$$

$$\frac{x^m}{y^n}+\frac{y^n}{x^m} = (\cos(m\theta - n\varphi) + i\sin(m\theta - n\varphi))+(\cos(m\theta - n\varphi) - i\sin(m\theta - n\varphi))$$

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\varphi)$$

$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2\sin(m\theta - n\varphi)$

(9) If $x = \cos\alpha + i\sin\alpha$; $y = \cos\beta + i\sin\beta$ prove that

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

$x = \cos\alpha + i\sin\alpha$

$y = \cos\beta + i\sin\beta$

$x^m = (\cos\alpha + i\sin\alpha)^m = (\cos m\alpha + i\sin m\alpha)$

$y^n = (\cos\beta + i\sin\beta)^n = (\cos n\beta + i\sin n\beta)$

$x^m . y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$

$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$

$x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) + \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

(10) If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$ and $c = \cos 2\gamma + i\sin 2\gamma$ prove that

(i) $abc + 1/\sqrt{abc} = 2\cos(\alpha + \beta + \gamma)$

(ii) $a^2b^2 + c^2/\sqrt{abc} = 2\cos 2(\alpha + \beta - \gamma)$

solution :

abc = (cos2α + i sin 2α) (cos2β + i sin 2β)(cos 2γ + i sin 2γ)

$\sqrt{abc}$ =

$$\sqrt{[(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma)]^{\frac{1}{2}}}$$

= (cosα + i sin α) (cosβ + i sin β)(cos γ + i sin γ)

= cos (α+β +γ) + i sin (α+β +γ)

$\frac{1}{\sqrt{abc}}$= cos (α+β +γ) - i sin (α+β +γ)

$$\sqrt{abc} + \frac{1}{\sqrt{abc}}$$

= cos (α+β +γ) + i sin (α+β +γ)+ cos (α+β +γ) + i sin (α+β +γ)

= 2 cos (α+β +γ) .

(ii) $a^2b^2 + c^2/\sqrt{abc} = \frac{a^2b^2}{abc} + \frac{c^2}{abc} = \frac{ab}{c} + \frac{c}{ab}$

$$\frac{ab}{c} = \frac{(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)}{(\cos 2\gamma + i\sin 2\gamma)}$$

= cos (2α+2β -2 γ) + i sin (2α+2β -2γ)

$$\frac{c}{ab} = \frac{1}{\frac{ab}{c}} = \frac{1}{\cos(2\alpha + 2\beta - 2\gamma) + i\sin(2\alpha + 2\beta - 2\gamma)}$$

= cos (2α+2β -2 γ) + i sin (2α+2β -2γ)

$\frac{ab}{c} + \frac{c}{ab}$ = cos (2α+2β -2 γ) + i sin (2α+2β -2γ) +cos (2α+2β -2 γ) - i sin (2α+2β -2γ)

= 2 cos (2α+2β -2 γ)

$a^2b^2 + c^2/\sqrt{abc}$ = 2 cos (2α+2β -2 γ) .

EXERCISE : 3 .5

find the value of the following :

$$(i)^{\frac{1}{3}} \quad (8i)^{\frac{1}{3}} \left(-\sqrt{3-i}\right)^{\frac{2}{3}}$$

(i) solution : we know that i = $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$(i)^{\frac{1}{3}} = \left(cos\frac{\pi}{2} + isn\frac{\pi}{2}\right)^{\frac{1}{3}}$$

$$= \left(cos\left(2k\pi + \frac{\pi}{2}\right) + isn\left(2k\pi + \frac{\pi}{2}\right)\right)^{\frac{1}{3}}$$

$$= \left(cos(4k+1)\frac{\pi}{2} + isn(4k+1)\frac{\pi}{2}\right)^{\frac{1}{3}}$$

$$= \left(cos(4k+1)\frac{\pi}{6} + isn(4k+1)\frac{\pi}{6}\right)^{1}$$

k = 0 , 1 , 2 …….

the value are cis$\frac{\pi}{6}$, cis$\frac{5\pi}{6}$, cis$\frac{9\pi}{6}$

(iii) solution : we know that 8 i = $8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$$(8i)^{\frac{1}{3}} = (8)^{\frac{1}{3}} \left(cos\frac{\pi}{2} + isn\frac{\pi}{2}\right)^{\frac{1}{3}}$$

$$= 2\left(cos\left(2k\pi + \frac{\pi}{2}\right) + isn\left(2k\pi + \frac{\pi}{2}\right)\right)^{\frac{1}{3}}$$

$= \; 2(cos(4k+1)\frac{\pi}{2} + \; isn(4k+1)\frac{\pi}{2})^{\frac{1}{3}}$

$= \; 2(cos(4k+1)\frac{\pi}{6} + \; isn(4k+1)\frac{\pi}{6})^{1}$

k = 0 , 1 , 2 …….

the value are 2cis$\frac{\pi}{6}$, 2cis$\frac{5\pi}{6}$, 2cis$\frac{9\pi}{6}$

(iii) let r = z

= -$\sqrt{3}$ –i

= $\sqrt{3+1}$ = 2

α = $\tan^{-1}\frac{b}{a}$

= $\tan^{-1}\frac{-1}{-\sqrt{3}}$ = $\tan^{-1}\frac{1}{\sqrt{3}}$

= $\frac{\pi}{6}$

$\theta = -\pi + \frac{\pi}{6}$ = $\frac{-5\pi}{6}$

polar form : z = r (cos $\theta + isin\theta$)

z = 2 (cos$\frac{-5\pi}{6}$ $+ isin\frac{-5\pi}{6}$)

$(-\sqrt{3} - i)^{\frac{2}{3}} = \; 2^{\frac{2}{3}}(cos\frac{-5\pi}{6} + \; isn\frac{-5\pi}{6})^{\frac{2}{3}}$

$= \; 2^{\frac{2}{3}}$(cos$(2k\pi - \frac{5\pi}{6})$ $+ isin(2k\pi - \frac{5\pi}{6}$ $=$

$2^{\frac{2}{3}}$(cos$(12k-5)\frac{\pi}{9}$ $+ isin(12k-5)\frac{\pi}{9}$)

k = 0 , 1 , 2 …….

the value are $2^{\frac{2}{3}}$cis$\frac{-5\pi}{9}$, $2^{\frac{2}{3}}$ cis$\frac{7\pi}{9}$, $2^{\frac{2}{3}}$ cis$\frac{19\pi}{9}$

(2) If $x = a + b$, $y = a\omega + b\omega^2$, $z = aw^2 + b\omega$ show that

(i) $xyz = a^3 + b^3$

(ii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ where ω is the complex cube root of unity.

solution: (i) xyz = (a+b)(a$\omega+b\omega^2$)(a$\omega^2 + b\omega$)

= (a+b)($a^2 + ab\omega^2 + ab\omega + b^2$)

= (a+b)($a^2 + ab(\omega^2 + \omega) + b^2$) **($(\omega^2 + \omega) = -1$)**

= (a+b)($a^2 - ab + b^2$)

= $a^3 + b^3$

(ii) x +y +z = (a+b)+(a$\omega+b\omega^2$)+(a$\omega^2 + b\omega$)

= a$(\omega^2 + \omega + 1)$ +b$(\omega^2 + \omega + 1)$

= a(0) +b(0) = 0

(3) Prove that if $\omega^3 = 1$, then

(i) $(a + b + c)\ (a + b\omega + c\omega^2)\ (a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$

(ii) $\left(\frac{-1 + i\sqrt{3}}{2}\right)^5 + \left(\frac{-1 - i\sqrt{3}}{2}\right)^5 = -1$

(iii) $\frac{1}{1 + 2\omega} - \frac{1}{1 + \omega} + \frac{1}{2 + \omega} = 0$

solution :

(i) (a+ b+ c) (a +bω +cω^2)(a +b$\omega^2 + c\omega$)

= (a+ b+ c) (a^2+ abω^2+ ac ω + ab ω + b^2 + bcω^2 +acω^2 + bc ω +c^2)

= (a+ b+ c) (a^2+ b^2 + c^2 + ab$(\omega^2 + \omega)$ + bc$(\omega + \omega^2)$ +ac$(\omega^2 + \omega)$)

= (a+ b+ c) (a^2+ b^2 + c^2 -2 ab -2 bc -2ac)

$= a^3 + b^3 + c^3$ -3abc

if ω is a cube root of unity , then $\omega = \frac{-1+i\sqrt{3}}{2}$ $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

$\therefore \left(\frac{-1+i\sqrt{3}}{2}\right)^5 + \left(\frac{-1-i\sqrt{3}}{2}\right)^5 = \omega^5 + (\omega^2)^5$

$= \omega^5 + \omega^{10}$

$= \omega^2 + \omega = -1$

(iii) L.H.S . $= \frac{1}{1+2\omega} - \frac{1}{1+\omega} + \frac{1}{2+\omega}$

$\frac{1}{1+2\omega} + \frac{1}{2+\omega} = \frac{2+\omega+1+2\omega}{(1+2\omega)(2+\omega)}$

$= \frac{3\,(1+\omega)}{2\omega^2+5\omega+2}$

$= \frac{3\,(1+\omega)}{2(\omega^2+\omega+1)+3\omega} = \frac{1+\omega}{\omega} = \frac{-\omega^2}{\omega}$

$= -\omega$

Again $\frac{1}{1+2\omega} = \frac{1}{\omega^2} = \frac{\omega^2}{\omega^2} = -\omega$

LHs = $-\omega\,(-\omega)$= 0 = R.H.S

4) Solve : (i) $x^4+4=0$ (ii) $x^4+x^3+x^2-x+1=0$

SOLUTION : $x^4+4=0$

$x^4 = -4 = 4\,(-1)$

$x = 4^{\frac{1}{4}}\,(-1)^{\frac{1}{4}}$

$= (2^2)^{\frac{1}{4}}\,[\cos\pi + i\sin\pi]^{\frac{1}{4}}$ sin ce $-1 = cis\,\pi$

$= 2^{\frac{1}{2}}\,[\cos(2k\pi+\pi) + i\sin(2k\pi+1)]^{\frac{1}{4}}$

$=\sqrt{2}\left[\cos(2k+1)\frac{\pi}{4}+i\sin(2k+1)\frac{\pi}{4}\right], k=0,1,2,3$

The values are $\sqrt{2}\text{cis}\,\frac{\pi}{4}$, $\sqrt{2}$ cis $\frac{3\pi}{4}$, $\sqrt{2}\text{cis}\,\frac{5\pi}{4}$, $\sqrt{2}$ cis $\frac{7\pi}{4}$

(ii) $x^4-x^3+x^2-x+1=0$

The terms of the polynomial are in GP with r = -x, a =1, n=5

$1-x+x^2-x^3+x^4 = \frac{a(r^n-1)}{r-1} = \frac{x^5+1}{r+1}$ where $x \neq -1$

Solve $x^5+1=0$ and remove the root x = -1

$x^5+1 = 0 \Rightarrow x = (-1)^{\frac{1}{5}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{5}}$

$= [(\cos(2k\pi+\pi + i\sin(2k\pi+\pi)]^{\frac{1}{5}}$

$= \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}$, where k = 0, 1,2,3,4

The values are cis $\frac{\pi}{5}$, cis $\frac{3\pi}{5}$, cis π, cis $\frac{7\pi}{5}$, cis $\frac{9\pi}{5}$

Remove the root cis π =-1

The roots are cis $\frac{\pi}{5}$, cis $\frac{3\pi}{5}$, cis $\frac{7\pi}{5}$, cis $\frac{9\pi}{5}$

5. Find all the values of $\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$ and hence prove that the product of the values is 1

SOLUTION : Let $\frac{1}{2}-i\frac{\sqrt{3}}{2}$ = r (cos θ + i sin θ)

r cos θ= $\frac{1}{2}$, sin θ = - $\frac{\sqrt{3}}{2}$

$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

Also cos θ = $\frac{1}{2}$, sin θ = - $\frac{\sqrt{3}}{2}$ θ in the 4th quadrant

$$\theta = -\frac{\pi}{3}$$

$\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{2}$

$$\frac{1}{2} - i\frac{\sqrt{3}}{2} = \left[cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]^{\frac{1}{4}}$$

$$= [\cos(-\pi) + i\sin(-\pi)]^{\frac{1}{4}}$$

$$= [\cos(2k\pi - \pi) + i\sin(2k\pi - \pi)]$$

$$= \cos\left(\frac{2k-1}{4}\right)\pi + i\sin\left(\frac{2k-1}{4}\right)\pi,$$

k =0,1,2,3

The values are $cis\left(-\frac{\pi}{4}\right), cis\left(\frac{\pi}{4}\right), cis\left(\frac{3\pi}{4}\right), cis\left(\frac{5\pi}{4}\right)$

Product of these values is $cis\left(-\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right) + \left(\frac{3\pi}{4}\right) + \left(\frac{5\pi}{4}\right)$

$$= cis\left[\frac{8\pi}{4}\right] = cis\ 2\pi$$

$$= \cos 2\pi + i\sin 2\pi$$

$$= 1$$

4.Analytical Geometry

Example sums:

1. Find the equation of the following parabola with indicated focus and directrix,

(i) (a,0) ; x= -a a>0

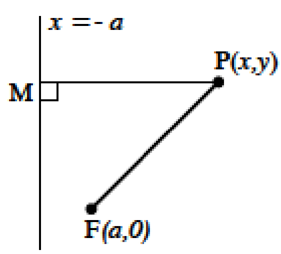
(ii) (-1,-2) ; x-2y+3=0

(iii) (2,-3) ; y-2=0

Solution:

(i) let P(x, y) be any point on the parabola. If PM is drawn perpendicular to the directrix,

$$\frac{FP}{PM}=e=1$$



$$FP^2=PM^2$$

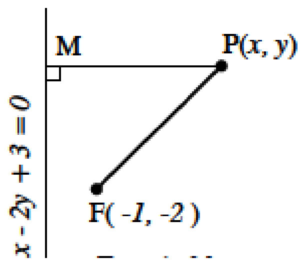
$$(x-a)^2+(y-0)^2=\left(\pm\frac{x+a}{\sqrt{1^2}}\right)^2$$

$$(x-a)^2+y^2=(x+a)^2$$

$$y^2= 4ax$$

This is the required equation.

(ii)) let P(x, y) be any point on the parabola. If PM is drawn perpendicular to the directrix,



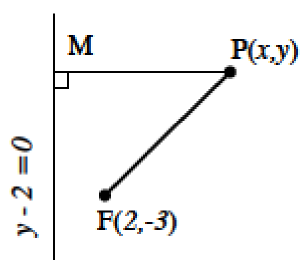
$$\frac{FP}{PM}=e=1$$

$$FP^2=PM^2$$

$$(x+1)^2+(y+2)^2=\left(\pm\frac{x-2y+3}{\sqrt{1^2+2^2}}\right)^2$$

$\Rightarrow 4x^2+4xy+y^2+4x+32y+16=0$

(iii) let P(x,y) be any point on the parabola. If PM is drawn perpendicular to the directrix,



$$\frac{FP}{PM}=e=1$$

$$FP^2=PM^2$$

i.e., $(x-2)^2+(y+3)^2=\left(\pm\frac{y-2}{\sqrt{1^2}}\right)^2$

$$(x-2)^2+(y+3)^2=(y-2)^2$$

⇨ $X^2-4x+10y+9=0$

2. Find the equation of the parabola if

(i) the vertex is (0,0) and focus is (-a,0), a>0

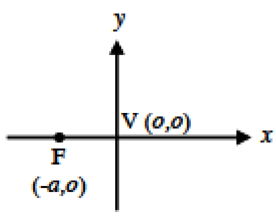
(ii) the vertex is (4,1) and focus is (4,-3)

Solution:

(i)From the given data the parabola is open leftward

The equation of the parabola is of the form

$$(y-k)^2 = -4a(x-h)$$



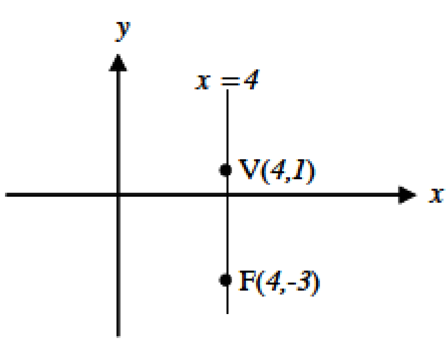
Here, the vertex (h, k) is (o, o) and VF=a

∴The required equation is

$$(y-0)^2 = -4a(x-0)$$

$$Y^2 = -4ax$$

(ii) From the given data the parabola is open downward.



The equation of the parabola is of the form

$(x-h)^2 = -4a(y-k)$

Here, the vertex (h, k) is (4,1) and

VF=a

$\Rightarrow \sqrt{(4-4)^2+(1+3)^2} = 4 = a$

∴The required equation is

$(x-4)^2 = -4\,(4)\,(y-1)$

$(x-4)^2 = -16\,(y-1)$

3. Find the equation of the parabola whose vertex is (1,2) and the equation of the latus rectum is x=3 .

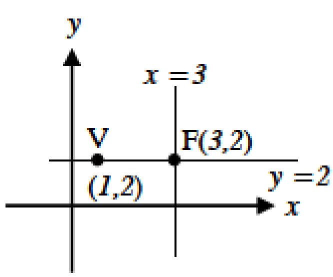
Solution:

From the given data the parabola is open rightward

The equation of the parabola is of the form

$(y-k)^2 = 4a(x-h)$

Here, the vertex (h, k) is (1,2)



Draw a perpendicular from V to the latus rectum.

It passes through the focus.

∴F is (3,2)

Again VF=a=2

∴The required equation is

$$(y-2)^2= 4 (2) (x-1)$$

$$(y-2)^2= 8 (x-1)$$

4. Find the equation of the parabola if the curve is open rightward, vertex is (2, 1) and passing through point (6, 5).

Solution:

Since it is open rightward, The equation of the parabola is of the form

$$(y- k)^2= 4a(x-h)$$

The vertex V (h, k) is (2,1)

$$(y- 1)^2= 4a(x- 2)$$

But it passes through (6,5)

$$4^2= 4 a (6-2)$$

$$a=1$$

∴The required equation is

$$(y- 1)^2= 4(x- 2)$$

5. Find the equation of the parabola if the curve is open upward, vertex is (-1,-2) and the length of the latus rectum is 4.

Solution:

Since it is open upward, The equation of the parabola is of the form

$(x- h)^2= 4a(y- k)$

Length of the latus rectum=4a=4 and this gives a=1

The vertex V (h, k) is (-1, -2)

Thus the required equation becomes

$(x+ 1)^2= 4(y+ 2)$

6. Find the equation of the parabola if the curve is open leftward, vertex is (2, 0) and the distance between the latus rectum and directrix is 2.

Solution:

Since it open leftward, the equation is of the form

$(y- k)^2 = - 4 a (x- h)$

The vertex V (h, k) is (2,0)

The distance between the latus rectum and directrix=2a= 2

Giving a=1 and

The equation of the parabola is

$(y- 0)^2= - 4 a (x- h)$

(0r)

$Y^2= - 4 (x- 2)$

7. Find the axis, vertex, focus, directrix, equation of the latus rectum, Length of the latus rectum for the following parabolas and hence draw their graphs.

(i) $y^2 = 4x$ (ii) $x^2 = -4y$ (iii) $(y+2)^2 = -8(x+1)$

(iv) $y^2 - 8x + 6y + 9 = 0$ (v) $x^2 - 2x + 8y + 17 = 0$

Solution:

`(i) $y^2 = 4x$

$(y-0)^2 = 4(1)(x-0)$

Here, (h, k) is (0, 0) and a=1

Axis : The axis of symmetry is x-axis.

Vertex : The vertex V (h, k) is (0, 0)

Focus : The focus F(a, 0) is (1, 0)

Directrix : The equation of the directrix is $x = -a$

i.e., $x = -1$

Latus

Rectum : The equation of the latus rectum is

$X = a$ i.e., $x = 1$

Length : $4a = 4(1) = 4$

The graph of the parabola looks as in fig.

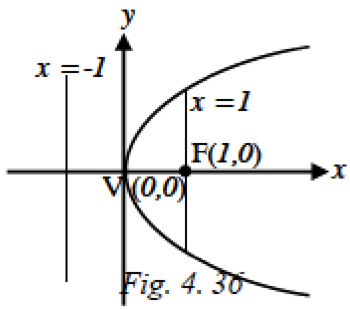


Fig. 4. 36

(ii) $x^2 = -4y$

$(x-0)^2 = -4(1)(y-0)$

Here, (h, k) is (0, 0) and a=1

Axis : The axis of symmetry is y-axis. Or x=0

Vertex : The vertex V (h, k) is (0, 0)

Focus : The focus F(0, -a) is (0, -1)

Directrix : The equation of the directrix is y= a

i.e., y = 1

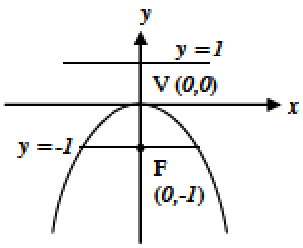
Latus

Rectum : The equation of the latus rectum is

y= - a i.e., y = -1

Length : 4a= 4(1)=4

The graph of the parabola looks as in fig.



(iii) $(y+2)^2 = -8(x+1)$

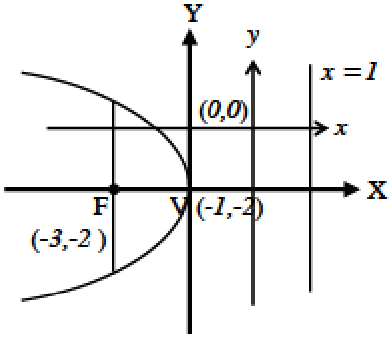
$Y^2 = -8X$ where $X = x+1$

$Y = y+2$

$Y^2 = -4(2)X$ $a=2$

The type is open leftward.

	Referred to X, Y	Referred to x, y X= x+1, Y= y+2
Axis	Y=0	Y=0 => y+2=o
Vertex	(0, 0)	X=0 ; Y=0 ⇨ x+1=0 ; y+2=0 ⇨ x= -1 ; y= -2 V (-1, -2)
Focus	(-a, 0) i.e., (-2, 0)	X= -2 ; Y=0 ⇨ x+1= -2; y+2=0 ⇨ x= -3 ; y= -2 F (-3, -2)
Directrix	X=a i.e., X=2	X= 2 ⇨ x+1=2 ⇨ x=1
Latus rectum	X= - a i.e. X= -2	X= -2 ⇨ X+1= -2 ⇨ X= -3
Length of latus rectum	4a= 8	8

Fig: 

iv) y^2 -8x+6y+9=0

y^2+6y=8x-9

$(y+3)^2$-9=8x-9

$(y+3)^2$=8x

Y^2=8X, a=2 where X=x

Y=y+3

The type is open rightward.

	Referred to X, Y	Referred to x, y X= x, Y= y+3
Axis	Y=0	Y=0 => y+3=o
Vertex	(0, 0)	X=0 ; Y=0 ⇨ x=0 ; y+3=0 ∴V (0, -3)
Focus	(a, 0) i.e., (2, 0)	X= 2 ; Y=0 ⇨ x= 2 ; y+3= 0 F (2, -3)
Directrix	X= -a i.e., X= -2	X= -2 ⇨ x=-2
Latus rectum	X= a i.e. X= 2	X= 2 ⇨ x= 2
Length of latus rectum	4a= 8	8

(v) x^2-2x+8y+17=0

x^2-2x= -8y-17

$(x-1)^2$-1= -8y-17

$(x-1)^2$ = -8y-16

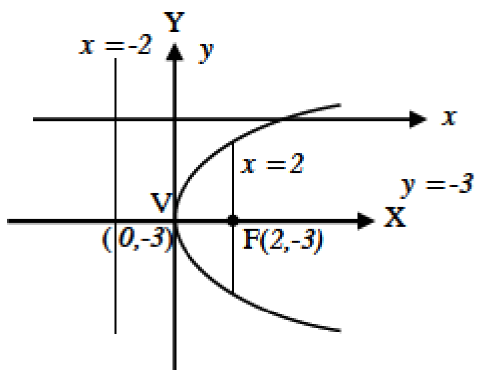
$(x-1)^2$ = -8 (y+2)

X^2= -8Y where X=x+1

X^2= -4(2) Y a=2 Y=y+2

The type is open downward.

Fig:



	Referred to X, Y	Referred to x, y X= x-1, Y= y+2
Axis	X=0	X=0 ⇨ X-1=o ⇨ X=1
Vertex	(0, 0)	X=0 ; Y=0 ⇨ X-1=0 ; y+2=0 ∴V (1, -2)

Focus	(0, -a) i.e., (0, -2)	X= 0 ; Y=-2 ⇨ X-1= 0 ; y+2= 0 F (1, -4)
Directrix	Y= a i.e., Y= 2	Y= 2 ⇨ Y+2=2 ⇨ Y=0
Latus rectum	Y= - a i.e. Y= -2	Y= -2 ⇨ Y+2= -2 ⇨ Y= -4
Length of latus rectum	4a= 8	8

8. The girder of a railway bridge is in the parabolic form with span 100 ft. And the highest point on the arch is 10 ft. above the bridge. Find the height of the bridge at 10 ft. to the left or from the midpoint Of the bridge.

Solution:

Consider the parabolic girder as open downwards

i.e., $x^2 = -4ay$

It passes through (50, -10)

$$50 \times 50 = -4a(-10)$$

$$a = \frac{250}{4}$$

$$x^2 = -4\left(\frac{250}{4}\right)y$$

$$x^2 = -250y$$

Let $B(10, y_1)$ be a point on the parabola.

$\therefore 100 = -250\ y_1$

$y_1 = -\frac{100}{250} = -\frac{2}{5}$

let AB be the height of the bridge at 10 ft to the right from the midpoint AC = $10 - \frac{2}{5} = 9\frac{3}{5}$ ft

i.e. the height of the bridge at the required place = $9\frac{3}{5}$ ft.

9. The headlight of a motor vehicle is a parabolic reflector of diameter 12cm and depth 4cm. find the position of bulb on the axis of the reflector for effective functioning of the headlight.

Solution:

By the property of parabolic reflector the position of the bulb should be placed at the focus.

By taking the vertex at the origin, the equation of the reflector is

$$Y^2 = 4ax$$

Let PQ be the diameter of the reflector

$\therefore P\ is\ (4,6)\ and\ since\ \ P(4,6) lies\ on\ parabola,$

36 = 4ax4

⇨ a = 2.25

The focus is at a distance of 2.25cm from the vertex

On the x-axis.

$\therefore The\ bulb\ has\ to\ be\ placed\ at\ a\ distance\ of\ 2.25 cms\ from$

the centre of the mirror.

10. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4mts when it is 6mts away from the point of projection. Finally it reaches the ground 12mts away from the starting point. Find the angle of projection.

Solution:

The equation of the parabola is of the form

$x^2 = -4ay$

it passes through(6, -4)

36 = 16a

⇨ $a = \frac{9}{4}$ ------- (1)

The equation is $x^2 = -9y$

Find the slope at (-6, -4)

Differentiating (1) with respect to x, we get

$2x = -9\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{2}{9}x$

At (-6, -4), $\frac{dy}{dx} = -\frac{2}{9}(-6) = \frac{4}{3}$

i.e. $\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

Thus angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$

11. A reflecting telescope has a parabolic mirror for which the distance from the vertex to the focus is 9mts. If the distance across (diameter) the top of the mirror is 160cm, how deep is the mirror at the middle?

Solution:

Let the vertex be at the origin.

$$VF = a = 900$$

The equation of the parabola is

$$y2 = 4 \times 900 \times x$$

Let x_1 be the depth of the mirror at the middle

Since $(x_1, 80)$ lies on the parabola

$$80^2 = 4 \times 900 \times x_1 \Rightarrow x_1 = \frac{16}{9}$$

$\therefore$ Depth of the mirror = $\frac{16}{9}$ cm.

12. Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:

As per the given information, we can take the parabola as open downwards

i.e. $x^2 = -4ay$

Let P be the point on the flow path, 2.5m below the line of the pipe and 3m beyond the vertical line through the end of the pipe.

$\therefore P$ is $(3, -2.5)$

Thus $9 = -4a\ (-2.5)$

$\Rightarrow a = \frac{9}{10}$

$\therefore$ The equation of the parabola is $x^2 = -4 \times \frac{9}{10} y$

Let $x1$ be the distance between the bottom of the vertical line on the ground from the pipe end and the point on which the water touches the ground. But the height of the pipe from the ground is 7.5 m

The point $(x_1, -7.5)$ lies on the parabola

$x_1^2 = -4 \times \frac{9}{10} \times (-7.5) = 27$

$x_1 = 3\sqrt{3}$

$\therefore$ The water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

13. A comet is moving in a parabolic orbit around the sun which is at the focus of a parabola. When the comet is 80 million kms from the sun, the line segment from the sun to the comet makes an angle of $\frac{\pi}{3}$ radians with the axis of the orbit. find (i) the equation of the comet's orbit (ii) how close does the comet comet come nearer to the sun? (Take the orbit as open rightward).

Solution:

Take the parabolic orbit as open rightward and the vertex at the origin.

Let P be the position of the comet in

Which FP = 80 million kms.

Draw a perpendicular PQ from P to the
Axis of the parabola.

Let $FQ = x_1$

From the triangle FQP,

$$PQ = FP \,.\, \sin\frac{\pi}{3}$$

$= 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3}$

Thus $FQ = x_1 = FP \,.\, \cos\frac{\pi}{3}$

$= 80 \times \frac{1}{2} = 40$

$\therefore VQ = a + 40$ if $VF = a$

P is $(VQ, PQ) = (a + 40,\ 40\sqrt{3})$

Since P lies on the parabola

$$y^2 = 4ax$$

$(40\sqrt{3})^2 = 4a\,(a + 40)$

$\Rightarrow a = -60$ or 20

$a = -60$ is not acceptable.

$\therefore$ The equation of the orbit is

$y^2 = 4 \times 20 \times x$

$y^2 = 80x$

The shortest distance between the Sun and the Comet is VF i.e. a

$\therefore$ The shortest distance is 20 million kms.

14. A cable of a suspension bridge hangs in the form of a parabola when the load is uniformly distributed horizontally. The distance between two towers is 1500 ft, the points of support of the cable on the towers are 200ft above the road way and the lowest point on the cable is 70ft above the roadway. Find the vertical distance to the cable (parallel to the roadway) from a pole whose height
is 122 ft.

Solution :

Take the lowest point on the cable as the vertex and take it as origin.

Let AB and CD be the towers. Since the distance between the two towers is 1500 ft.

$VA' = 750$ ft ; $AB = 200$ ft

$\therefore A'B = 200 - 70 = 130$ ft

Thus the point B is (750, 130)

The equation of the parabola is $x^2 = 4ay$

Since B is a point on $x^2 = 4ay$

$$(750)^2 = 4a(130)$$

$$\Rightarrow 4a = \frac{75\times750}{13}$$

$$\therefore \text{The equation is } x^2 = \frac{75\times750}{13}y$$

Let PQ be the vertical distance to the cable from the pole RQ.

$RQ = 122$, $RR' = 70$

$\Rightarrow R'Q = 52$

Let VR be $x1$ $\therefore Q$ is $(x_1, 52)$

Q is a point on parabola

$$x_1^{\,2} = \frac{75\times750}{13}\times 52$$

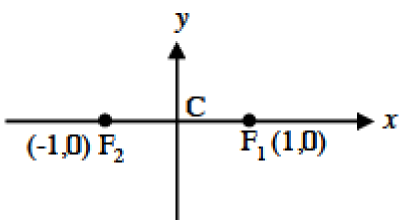
$$x_1 = 150\sqrt{10}$$

$$PQ = 2x_1 = 300\sqrt{10} \text{ ft.}$$

15. Find the equation of the ellipse whose foci are (1, 0) and (−1, 0) and eccentricity is $\frac{1}{2}$

Solution:

The centre of the ellipse is the midpoint of FF where F is (1, 0) and F' is (−1, 0).



$\therefore$ Centre C is $= \left(\frac{1-1}{2}, \frac{0+0}{2}\right)$

But $F_1F_2 = 2ae = 2$ and $e = \frac{1}{2}$

$$2a\times\frac{1}{2} = 2$$

⇨ $a = 2$

$$b^2 = a^2(1-e^2)$$

$$= 4\left(1-\frac{1}{4}\right)$$

$$= 3$$

From the given data the major axis is along x-axis.

$\therefore$ the equation of the ellipse is of the form

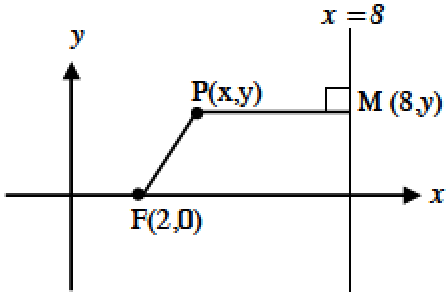
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1$

16. Find the equation of the ellipse whose one of the foci is (2, 0) and the corresponding directrix is $x = 8$ and eccentricity is $\frac{1}{2}$

Solution:

Let $P(x, y)$ be a moving point.



By definition $\frac{FP}{PM}$=e^2

$FP^2 = e^2\, PM^2$

$(x-2)^2+(y-0)^2 = \frac{1}{4}\left(\pm\frac{x-8}{\sqrt{1^2}}\right)^2$

$(x-2)^2+y^2 = \frac{1}{4}\ (x-8)^2$

$4\ [(x-2)^2+y^2] = (x-8)^2$

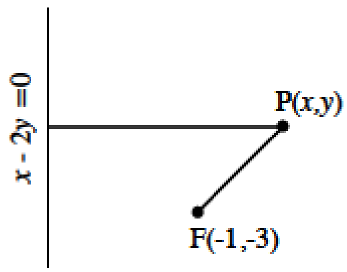
$3x^2+4y^2 = 48$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

17. find the equation of the ellipse with focus (-1,-3), directrix x-2y=0 and eccentricity $\frac{4}{5}$

Solution:

let P(x, y) be a moving point.



By definition $\frac{FP}{PM} = e^2$

$FP^2 = e^2 PM^2$

$(x+1)^2 + (y+3)^2 = \frac{16}{25}\left(\pm\frac{x-2y}{\sqrt{1+4}}\right)^2$

$125\,[(x+1)^2 + (y+3)^2] = 16(x-2y)^2$

⇨ $109x^2 + 64xy + 61y^2 + 250x + 750y + 1250 = 0$

18. find the equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$

Solution:

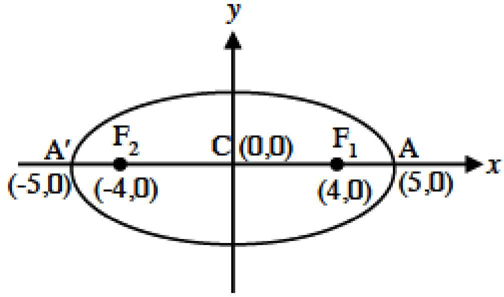
Let the foci be F_1 (4, 0) and

F_2 (-4, 0), vertices be A (5, 0) and A′ (-5, 0)

The centre is the midpoint of AA′

i.e., C is $\left(\frac{-5+5}{2}, \frac{0+0}{2}\right)$

C = (0, 0)



From the given data, the major axis is along the x-axis and the equation of the ellipse is of the form

$$\frac{(x)^2}{a^2}+\frac{(y)^2}{b^2}=1$$

Here $CA = a = 5$; $CF = ae = 4$

$b^2 = a^2 (1-e^2)$

$= 25-16 = 9$ and

The equation of the ellipse is

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

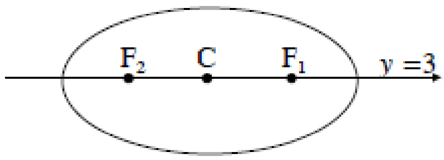
19. The centre of the ellipse is (2, 3). One of the foci is (3, 3). Find the other focus.

Solution:

From the given data the major axis is parallel to the x axis.

Let $F1$ be (3, 3)

Let $F2$ be the point (x, y).



Since C (2, 3) is the midpoint of $F1$ and $F2$ on the major axis $y = 3$

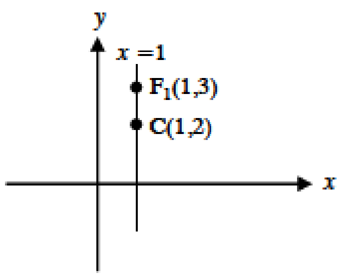
$$\frac{x+3}{2} = 2 \text{ and } \frac{y+3}{2} = 3$$

This gives x =1 and y=3.

Thus the other focus is (1, 3)

20. Find the equation of the ellipse whose centre is (1, 2), one of the foci is (1, 3) and eccentricity is $\frac{1}{2}$

Solution:



The major axis is parallel to y-axis.

∴The equation is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$CF_1 = ae = 1$

But $e = \frac{1}{2}$

⇨ $a = 2, \; a^2 = 4$

⇨ $b^2 = a^2 (1-e^2)$

$= 4\left(1 - \frac{1}{4}\right)$

$= 3$; C (h, k) = (1, 2)

Thus required equation is

$$\frac{(x-1)^2}{3} + \frac{(y-2)^2}{4} = 1$$

21. Find the equation of the ellipse whose major axis is along *x*-axis, centre at the origin, passes through the point (2, 1) and eccentricity $\frac{1}{2}$

Solution:

Since the major axis is along the *x*-axis and the centre is at the origin,

The equation of the ellipse is of the form

$$\frac{(x)^2}{a^2} + \frac{(y)^2}{b^2} = 1$$

It passes through the point (2, 1).

$$\therefore \frac{4}{a^2} + \frac{1}{b^2} = 1$$

$$e = \frac{1}{2}$$

⇨ $b^2 = a^2 (1-e^2)$

$= a^2\left(1 - \frac{1}{4}\right)$

$4b^2 = 3a^2$

Solving (1) and (2)

we get $a^2 = \frac{16}{3}$, $b^2 = 4$

Thus required equation is

$$\frac{(x)^2}{16/3} + \frac{(y)^2}{4} = 1$$

22. Find the equation of the ellipse if the major axis is parallel to *y*-axis, semi-major axis is 12, length of the latus rectum is 6 and the centre is (1,12)

Solution:

Since the major axis is parallel to *y*-axis

the equation of the ellipse is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

The centre $C(h, k)$ is (1, 12)

Semi major axis $a = 12 \Rightarrow a^2 = 144$

Length of the latus rectum $\frac{2b^2}{a} = 6$

$$\frac{2b^2}{12} = 6$$

$\therefore b^2 = 36$ and the required equation is

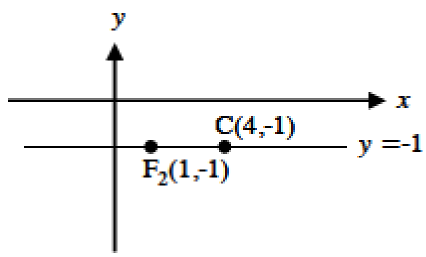
$$\frac{(x-1)^2}{36} + \frac{(y-12)^2}{144} = 1$$

23. Find the equation of the ellipse given that the centre is (4, −1), focus is (1, −1) and passing through (8, 0).

Solution:

From the given data since the

major axis is parallel to the *x* axis,



the equation is of the form

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

The centre $C(h, k)$ is $(4, -1)$

$$\frac{(x-4)^2}{a^2}+\frac{(y+1)^2}{b^2}=1$$

It passes through the point (8, 0)

$$\frac{(8-4)^2}{a^2}+\frac{(0+1)^2}{b^2}=1 \quad \text{----- (1)}$$

But $CF_1 = ae = 3$

⇨ $b^2 = a^2(1-e^2)$

$= a^2 - a^2e^2$

$= a^2 - 9$

⇨ (1) $\therefore \frac{16}{a^2}+\frac{1}{b^2}=1$

⇨ $\frac{16}{a^2}+\frac{1}{a^2-9}=1$

⇨ $16a^2-144+a^2 = a^4-9a^2$

⇨ $a^4-26a^2+144 = 0$

⇨ $a^2 = 8$; $a^2 = 18$

$b^2 = a^2-9$

(i) if $a^2 = 8$ then $b^2 = -1$

(ii) if $a^2 = 18$ then $b^2 = 9$

$$\therefore a^2 = 18; \quad b^2 = 9$$

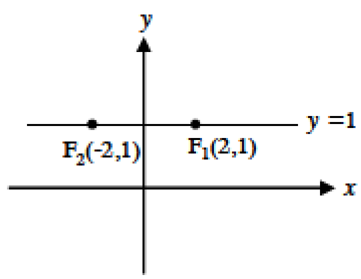
The required equation of the ellipse is

$$\frac{(x-4)^2}{18} + \frac{(y+1)^2}{9} = 1$$

24. Find the equation of the ellipse whose foci are (2, 1), (−2, 1) and length of the latus rectum is 6.

Solution:

From the given data the major axis is parallel to the *x* axis.



∴The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Since the centre is the midpoint of $F_1 F_2$

C is $\left(\frac{-2+2}{2}, \frac{1+1}{2}\right) = (0, 1)$

And the equation becomes

$$\frac{(x)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$$

$F_1 F_2 = 2ae = 4$

⇨ $a^2e^2 = 4$

⇨ $a^2e^2 = a^2 - b^2$

$\therefore a^2 - b^2 = 4$ -------- (1)

The length of the latus rectum

$$\frac{2b^2}{a} = 6$$

$b^2 = 3a$ ------- (2)

⇨ (1) $a^2 - 3a - 4 = 0$

⇨ $a = 4$ or -1

⇨ $a = -1$ is absurd

$a = 4$

$b^2 = 3a = 12$

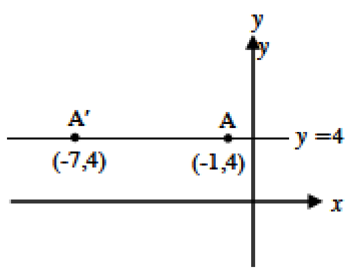
Thus the required equation is

$$\frac{(x)^2}{16} + \frac{(y-1)^2}{12} = 1$$

25. Find the equation of the ellipse whose vertices are (−1, 4) and (−7, 4) and eccentricity is $\frac{1}{3}$.

Solution:

From the given data the major axis is parallel to *x* axis.



$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The centre is the midpoint of AA'

$\therefore C$ is $\left(\frac{-1-7}{2}, \frac{4+4}{2}\right)$ = (-4, 4)

Thus the equation becomes

$$\frac{(x+4)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1$$

We know that

AA' = 2a = 6 => a = 3

$b^2 = a^2 (1-e^2) = 9\left(1 - \frac{1}{9}\right) = 8$

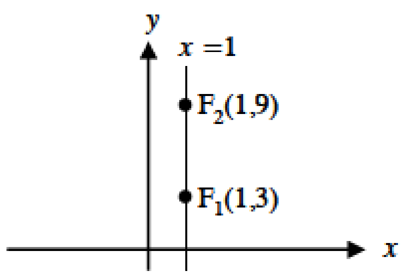
The required equation is

$$\frac{(x+4)^2}{9} + \frac{(y-4)^2}{8} = 1$$

26. Find the equation of the ellipse whose foci are (1, 3) and (1, 9) and eccentricity is $\frac{1}{2}$

Solution:

From the given data the major axis is parallel to *y* axis.



$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The centre of the ellipse is the midpoint of $F_1 F_2$

$\therefore C$ is $\left(\frac{1+1}{2}, \frac{3+9}{2}\right) = (1, 6)$

$F_1 F_2 = 2ae = 6$

$ae = 3$

but $e = \frac{1}{2}$; $\therefore a = 6$

$b^2 = a^2 (1-e^2) = 36\left(1 - \frac{1}{4}\right) = 27$

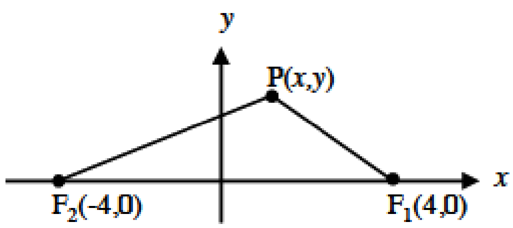
Thus the required equation is

$$\frac{(x-1)^2}{27} + \frac{(y-6)^2}{36} = 1$$

27. Find the equation of a point which moves so that the sum of its distances from (-4, 0) and (4, 0) is 10.

Solution:

Let F_1 and F_2 be the fixed points (4, 0) and (−4, 0) respectively and $P(x_1, y_1)$ be the moving point.



It is given that $F_1P + F_2P = 10$

$$\sqrt{(x_1-4)^2 + (y_1-0)^2} + \sqrt{(x_1+4)^2 + (y_1-0)^2} = 10$$

Simplifying we get

$9x_1^2 + 25y_1^2 = 225.$

$\therefore$ *the locus of* (x_1, y_1) *is*

$$\frac{(x)^2}{25} + \frac{(y)^2}{9} = 1$$

28. Find the equations and lengths of major and minor axes of

(i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (ii) $4x^2 + 3y^2 = 2$ (iii) $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$

Solution:

(i) The major axis is along *x*-axis and

the minor axis is along *y*-axis.

This gives the equation of major axis as $y = 0$ and

The equation of minor axis as $x = 0$.

We have $a^2 = 9$; $b^2 = 4$

$\Rightarrow a = 3$, $b = 2$

$\therefore$ The length of major axis is $2a = 6$ and

The length of minor axis is $2b = 4$

(ii) $4x^2 + 3y^2 = 2$

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

The major axis is along *y*-axis and the minor axis is along *x*-axis.

$\therefore$ The equation of major axis is $x = 0$ and

The equation of minor axis is $y = 0$.

Here $a^2 = 4$; $b^2 = 3$

$\Rightarrow a = 2$, $b = \sqrt{3}$

$\therefore$ The length of major axis $(2a) = 4$

The length of minor axis $(2b) = 2\sqrt{3}$

(iii) Let $x - 1 = X$ and $y + 1 = Y$

$\therefore$ The given equation becomes

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Clearly the major axis is along Y-axis and

the minor axis is along X-axis.

$\therefore$ The equation of major axis is $X = 0$ and

the equation of minor axis is $Y = 0$

i.e., the equation of major axis is $x - 1 = 0$ and

the equation of minor axis is $y + 1 = 0$

Here $a^2 = 16$, $b^2 = 9$

$\Rightarrow a = 4$, $b = 3$ $\therefore$ Length of major axis $(2a) = 8$

$\therefore$ Length of minor axis $(2b) = 6$

29. Find the equations of axes and length of axes of the ellipse

$6x^2 + 9y^2 + 12x - 36y - 12 = 0$

Solution:

$$6x^2 + 9y^2 + 12x - 36y - 12 = 0$$

$$(6x^2 + 12x) + (9y^2 - 36y) = 12$$

$$6(x^2 + 2x) + 9(y^2 - 4y) = 12$$

$$6\{(x + 1)^2 - 1\} + 9\{(y - 2)^2 - 4\} = 12$$

$$6(x + 1)^2 + 9(y - 2)^2 = 12 + 6 + 36$$

$$6(x+1)^2 + 9(y-2)^2 = 54$$

$$\frac{(x+1)^2}{9} + \frac{(y-1)^2}{6} = 1$$

Let X = x+1; Y = y-2

∴ The equation becomes

$$\frac{X^2}{9} + \frac{Y^2}{6} = 1$$

Clearly the major axis is along *X*-axis and

the minor axis is along *Y*-axis.

∴ The equation of the major axis is $Y = 0$ and

the equation of the minor axis is $X = 0$.

The equation of the major axis is $y - 2 = 0$ and of

minor axis is $x + 1 = 0$

i.e., the equation of the major axis is $y - 2 = 0$

Here $a^2 = 9$, $b^2 = 6$

$\Rightarrow a = 3$, $b = \sqrt{6}$

∴ The length of major axis $(2a) = 6$

The length of minor axis $(2b) = 2\sqrt{6}$

30. Find the equations of directrices, latus rectum and length of latus rectums of the following ellipses.

Solution:

(i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (ii) $25x^2 + 9y^2 = 225$

(iii) $4x^2 + 3y^2 + 8x + 12y + 4 = 0$

Solution :

(i) The major axis is along *x*-axis

Here $a^2 = 16, \quad b^2 = 9$

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

Equations of directrices are

$$x = \pm\frac{a}{e}$$

$$x = \pm\frac{16}{\sqrt{7}}$$

Equations of the latus rectums are

$$x = \pm\sqrt{ae}$$

$$x = \pm\sqrt{7}$$

length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2\times9}{4} = \frac{9}{2}$$

(ii) $25x^2 + 9y^2 = 225$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

Here $a^2 = 25, \quad b^2 = 9$

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Equations of directrices are

$$x = \pm\frac{a}{e}$$

$$x = \pm\frac{25}{4}$$

Equations of the latus rectums are

$$x = \pm ae$$

$$x = \pm 4$$

length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2\times 9}{5} = \frac{18}{5}$$

(iii) $4x^2 + 3y^2 + 8x + 12y + 4 = 0$

$$(4x^2 + 8x) + (3y^2 + 12y) + 4 = 0$$

$$4(x^2 + 2x) + 3(y^2 + 4y) = -4$$

$$4\{(x + 1)^2 - 1\} + 3\{(y + 2)^2 - 4\} = -4$$

$$4(x + 1)^2 + 3(y + 2)^2 = 12$$

$$\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$$

$\frac{X^2}{3} + \frac{Y^2}{4} = 1$ where $X = x-1$

$Y = y+1$

The major axis is along Y axis.

Here $a^2 = 4$, $b^2 = 3$ and $e = \frac{1}{2}$

Equations of the directrices are $Y = \pm\frac{a}{e}$

$$\text{i.e. } Y = \pm\frac{2}{(1/2)}$$

$$Y = \pm 4$$

(i) $Y = 4$

$y + 2 = 4$

$\Rightarrow y = 2$

(ii) $Y = -4$

$y + 2 = -4$

$\Rightarrow y = -6$

The directrices are $y = 2$ and $y = -6$

Equations of the latus rectum are $Y = \pm ae$

$$\text{i.e. } Y = \pm 2\left(\frac{1}{2}\right)$$

$$Y = \pm 1$$

(i) $Y = 1$

$\Rightarrow y + 2 = 1$

$\Rightarrow y = -1$

(ii) $Y = -1$

$\Rightarrow y + 2 = 1$

$\Rightarrow y = -3$

$\therefore$ Equation of the latus rectum are $y = -1$ and $y = -3$

Length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2\times 3}{2} = 3$$

31. Find the eccentricity, centre, foci, vertices, of the following ellipses:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (i) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (iii) $\frac{(x+3)^2}{6} + \frac{(y-5)^2}{4} = 1$

(iv) $36x^2+4y^2-72x+32y-44=0$

Solution:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

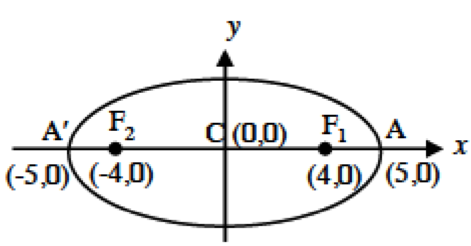
The major axis is along x-axis $a^2 = 25$, $b^2 = 9$

$e = \frac{4}{5}$ and $ae = 4$

Clearly centre C is (0, 0),

Foci are $(\pm ae, 0) = (\pm 4, 0)$

Vertices are $(\pm a, 0) = (\pm 5, 0)$



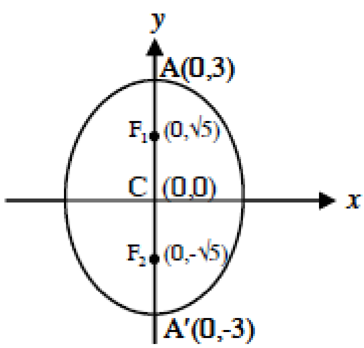
(ii) The major axis is along y-axis $a^2 = 9$, $b^2 = 4$

$e = \frac{\sqrt{5}}{3}$ and $ae = \sqrt{5}$

Clearly centre C is (0, 0)

Foci are $(0, \pm ae) = (0, \pm\sqrt{5})$

Vertices are $(0, \pm a) = (0, \pm 3)$



(iii) $\frac{(x+3)^2}{6} + \frac{(y-5)^2}{4} = 1$

Let $x + 3 = X$, $y - 5 = Y$

$\therefore$ The equation becomes

$$\frac{X^2}{6} + \frac{Y^2}{4} = 1$$

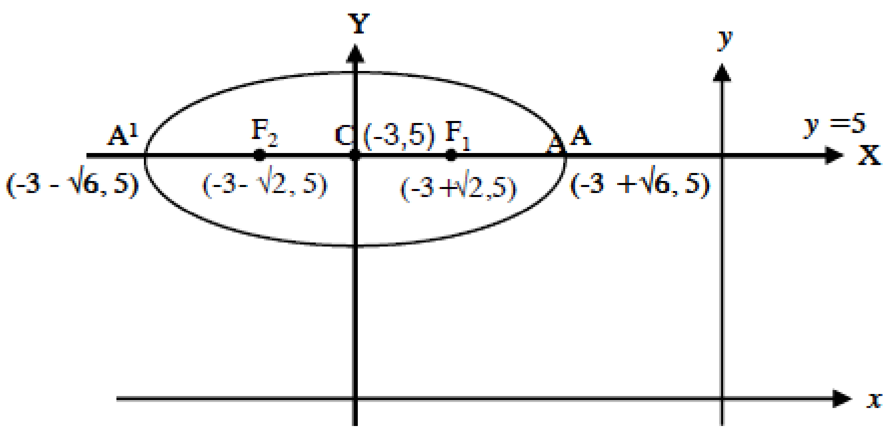
The major axis is along X-axis

$a^2 = 6$, $b^2 = 4$

$e = \frac{1}{\sqrt{3}}$ and $ae = \sqrt{2}$

	Referred to X, Y	Referred to x, y X= x+3, Y= y-5
Centre	(0, 0)	X=0 ; Y=0 ⇨ X+3=0 ; y-5=0 ∴C (-3, 5)

Vertices	$(\pm a, 0)$ i.e., $(\pm\sqrt{6}, 0)$ (i) $(\sqrt{6}, 0)$ (i) $(-\sqrt{6}, 0)$	(i) $X=\sqrt{6}$; $Y=0$ ⇨ $X+3=\sqrt{6}$; $y-5=0$ $X=\sqrt{6}-3$, $y=5$ $A(-3+\sqrt{6}, 5)$ (ii) $X=-\sqrt{6}$; $Y=0$ $X+3=-\sqrt{6}$ $y-5=0$ $X=-3-\sqrt{6}$, $y=5$ $A'(-3-\sqrt{6}, 5)$
Foci	$(\pm ae, 0)$ i.e., $(\pm\sqrt{2}, 0)$ (i) $(\sqrt{2}, 0)$ (ii) $(-\sqrt{2}, 0)$	(i) $X=\sqrt{2}$, $Y=0$ $X+3=\sqrt{2}$, $y-5=0$ $X=-3+\sqrt{2}$, $y=5$ $F_1(-3+\sqrt{2}, 5)$ (ii) $X=-\sqrt{2}$, $Y=0$ $X+3=-\sqrt{2}$, $y-5=0$ $X=-3-\sqrt{2}$, $y=5$ $F_2(-3-\sqrt{2}, 5)$



(iv) $36x^2+4y^2-72x+32y-44=0$

$36(x^2-2x)+4(y^2+8y)=44$

$36\{(x-1)^2-1\}+4\{(y+4)^2-16\}=44$

$36(x-1)^2 + 4(y+4)^2 = 144$

$$\frac{(x-1)^2}{4} + \frac{(y+4)^2}{36} = 1$$

$$\frac{X^2}{6} + \frac{Y^2}{4} = 1$$

where $X = x - 1$, $Y = y + 4$

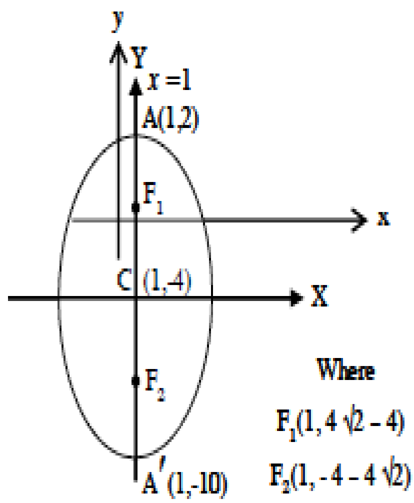
The major axis is along Y-axis.

$a^2 = 36$, $b^2 = 4$

$e = \frac{2\sqrt{2}}{3}$ and $ae = 4\sqrt{2}$

	Referred to X, Y	Referred to x, y X= x-1, Y= y+4
Centre	(0, 0)	X=0 ; Y=0 ⇨ X-1=0 ; y+4=0 ∴C (1, -4)
Vertices	$(0, \pm a)$ i.e., $(0, \pm 6)$ (i) (0, 6) (i) (0, -6)	(i) X= 0 ; Y=6 ⇨ X-1= 0 ; y+4=6 X =1, y =2 A (1, 2) (ii) X = 0; Y = -6 X-1 = 0, y+4 =-6 X = 1, y = -10 A′ (1, -10)
Foci	$(0, \pm ae)$ i.e., $(0, \pm 4\sqrt{2})$ (i) $(0, 4\sqrt{2})$	(i) X = 0 , Y = $4\sqrt{2}$ X-1 =0, y+4 = $4\sqrt{2}$ X = 1, y = $4\sqrt{2} - 4$ F_1 $(1, 4\sqrt{2} - 4)$

	(ii) $(0, -4\sqrt{2})$	(ii) $X = 0, Y = -4\sqrt{2}$ $x-1 = 0;\ y+4= -4\sqrt{2}$ $x =1; y = -4-4\sqrt{2}$ $F_2\ (1, -4-4\sqrt{2})$

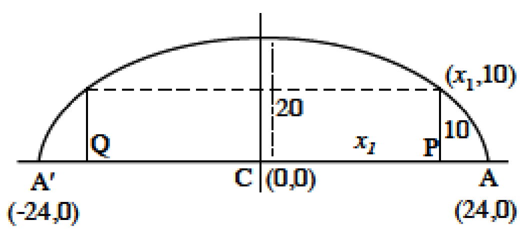


32. An arch is in the form of a semi-ellipse whose span is 48 feet wide. The height of the arch is 20 feet. How wide is the arch at a height of 10 feet above the base?

Solution:

Take the midpoint of the base as the centre $C\ (0, 0)$

Since the base wide is 48 feet,



The vertices A and A' are $(24, 0)$ and $(-24, 0)$ respectively.

Clearly $2a = 48$ and $b = 20$.

The corresponding equation is

$$\frac{x^2}{24^2} + \frac{y^2}{20^2} = 1$$

Let x_1 be the distance between the pole whose height is 10m and the centre.

Then $(x_1, 10)$ satisfies the equation (1)

$$\therefore \frac{x^2}{24^2} + \frac{y^2}{20^2} = 1$$

⇨ $x_1 = 12\sqrt{3}$

Clearly the width of the arch at a height of 10 feet is $2x_1 = 24\sqrt{3}$

Thus the required width of arch is $24\sqrt{3}$ feet.

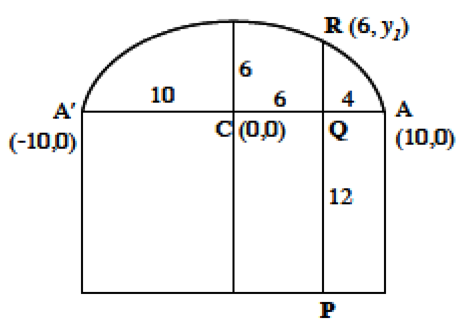
33. The ceiling in a hallway 20ft wide is in the shape of a semi ellipse and 18 ft high at the centre. Find the height of the ceiling 4 feet from either wall if the height of the side walls is 12ft.

Solution:

Let PQR be the height of the ceiling which is 4 feet from the wall.

From the diagram $PQ = 12$ ft

To find the height QR



Since the width is 20ft, take A, A' as vertices with A as (10, 0) and A' as (−10, 0).

Take the midpoint of AA' as the centre which is (0, 0)

From the diagram $AA' = 2a = 20$

$$\Rightarrow a = 10 \text{ and } b = 18 - 12 = 6$$

$$\frac{x^2}{100} + \frac{y^2}{36} = 1$$

Let QR be y_1 then R is $(6, y_1)$

Since R lies on the ellipse,

$$\frac{36}{100} + \frac{y_1^2}{36} = 1$$

$$\Rightarrow y_1 = 4.8$$

$\therefore PQ + QR = 12 + 4.8$

$\therefore$ The required height of the ceiling is 16.8 feet.

34. The orbit of the earth around the sun is elliptical in shape with sun at a focus. The semi major axis is of length 92.9 million miles and eccentricity is 0.017. Find how close the earth gets to sun and the greatest possible distance between the earth and the sun.

Solution :

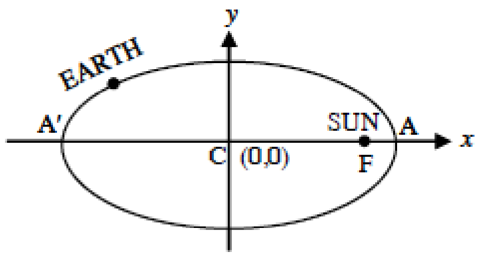
Semi-major axis CA is

$a = 92.9$ million miles

Given $e = 0.017$

The closest distance of the earth from the sun = FA

and farthest distance of the earth from the sun = FA'



$CF = ae = 92.9 \times 0.017$

$FA = CA - CF = 92.9 - 92.9 \times 0.017$

$= 92.9\ [1 - 0.017]$

$= 92.9 \times 0.983 = 91.3207$ million miles

$FA' = CA + CF = 92.9 + 92.9 \times 0.017$

$= 92.9\ (1 + 0.017)$

$= 92.9 \times 1.017 = 94.4793$ million miles

35. A ladder of length 15m moves with its ends always touching the vertical wall and the horizontal floor. Determine the equation of the locus of a point P on the ladder, which is 6m from the end of the ladder in contact with the floor.

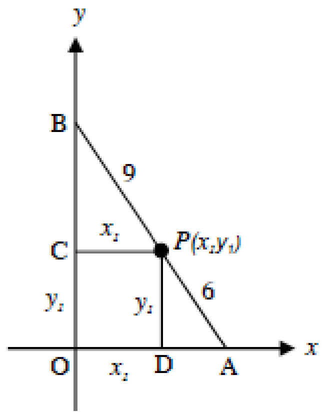
Solution:

Let AB be the ladder and $P(x_1, y_1)$ be a point on the ladder such that AP = 6m.

Draw PD perpendicular to x-axis and

PC perpendicular to y-axis.

Clearly the triangles ADP and PCB are similar.



$$\therefore \frac{PC}{DA} = \frac{PB}{AP} = \frac{BC}{PD}$$

i.e., $\frac{x_1}{DA} = \frac{9}{6} = \frac{BC}{y_1}$

=> $$DA = \frac{6x_1}{9} = \frac{2x_1}{3}$$

$$BC = \frac{9y_1}{6} = \frac{3y_1}{2}$$

$$OA = OD + DA = x_1 + \frac{2x_1}{3} = \frac{5x_1}{3}$$

$$OB = OC + BC = y_1 + \frac{3y_1}{2} = \frac{5y_1}{2}$$

But $OA^2 + OB^2 = AB^2$

⇨ $$\frac{25}{9}x_1^2 + \frac{25}{4}y_1^2 = 225$$

$$\boldsymbol{\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1}$$

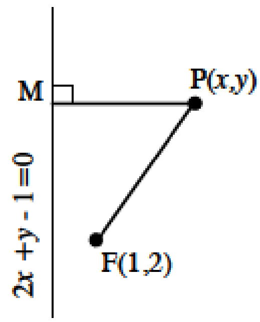
∴ The locus of (x_1, y_1) is

$\frac{x^2}{81} + \frac{y^2}{36} = 1$, which is an ellipse.

36. Find the equation of hyperbola whose directrix is 2x+y = 1, Focus (1, 2) and eccentricity $\sqrt{3}$.

Solution:

Led P(x, y) be any point on the hyperbola.



Draw PM perpendicular to the directrix.

By the definition,

$$\frac{FP}{PM} = e$$

$$FP^2 = e^2 PM^2$$

$$(x-1)^2+(y-2)^2 = 3\left(\pm\frac{2x+y-1}{\sqrt{4+1}}\right)^2$$

$$[(x-1)^2 + (y-2)^2] = \frac{3}{5}(2x+y-1)^2$$

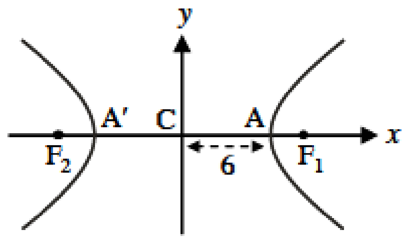
⇨ $7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$

Thus is the required equation of the hyperbola

37. Find the equation of the hyperbola whose transverse axis is along *x*-axis. The centre is (0, 0) length of semi-transverse axis is 6 and eccentricity is 3.

Solution:

Since the transverse axis is along *x*-axis and the centre is (0, 0),



The equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} =$$

Given that semi-transverse axis $a = 6$,eccentricity $e = 3$

We know that $b^2 = a^2\,(e^2 - 1)$

$$b^2 = 36\,(8)$$

$$= 288$$

∴The equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{288} = 1$$

38. Find the equation of the hyperbola whose transverse axis is parallel to *x*-axis, centre is (1, 2), length of the conjugate axis is 4 and eccentricity $e = 2$.

Solution:

Since the transverse axis is parallel to *x*-axis

The equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Here centre C(h, k)is (1, 2)

The length of conjugate axis 2b = 4 and e = 2

$b^2 = a^2 (e^2-1)$

$4 = a^2 (4-1)$

$a^2 = \frac{4}{3}$

Thus the required equation is

$$\frac{(x-1)^2}{4/3} - \frac{(y-2)^2}{4} = 1$$

39. Find the equation of the hyperbola whose centre is (1, 2).
The distance between the directrices is $\frac{20}{3}$, the distance between the foci is 30 and the transverse axis is parallel to *y*-axis.

Solution:

Since the transverse axis is parallel to *y*-axis, the equation is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Here centre *C*(*h*, *k*) is (1, 2)

The distance between the directrices

$$\frac{2a}{e} = \frac{20}{3} \Rightarrow \frac{a}{e} = \frac{10}{3}$$

The distance between the foci, $2ae = 30 \Rightarrow ae = 15$

$\frac{a}{e}(ae) = \frac{10}{3} \times 15$

$\Rightarrow a^2 = 50$

Also $\frac{ae}{a/e} \Rightarrow e^2 = \frac{9}{2}$

The required equation is

$b^2 = a^2 (e^2-1)$

$b2 = 50\,(\frac{9}{2} - 1)$

$= 175$

The required equation is

$$\frac{(y-2)^2}{50} - \frac{(x-1)^2}{175} = 1$$

40. Find the of equation the hyberbola whose transverse is parallel to y-axis is 4 and eccentricity is 2.

Solution:

From the given data the hyperbola is of form

$$\frac{(y)^2}{a^2} - \frac{(x)^2}{b^2} = 1$$

Given that semi- conjugate axis b = 4 and e = 2.

$b^2 = a^2\,(e^2-1)$

$4^2 = a^2\,(2^2-1)$

$a^2 = \frac{16}{3}$

Hence the equation of the hyperbola is

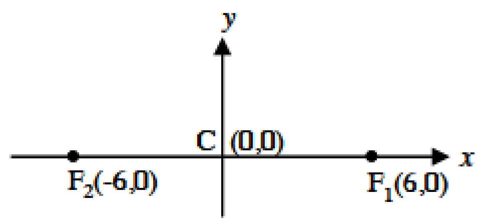
$$\frac{(y)^2}{16/3} - \frac{(x)^2}{4} = 1$$

$3y^2 - x^2 = 16$

41. Find the equation of the hyperbola whose foci are $(\pm 6, 0)$ and length of the transverse axis is 8.

Solution:

From the given data the transverse axis is along x-axis.



The equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The centre is the midpoint of F_1 and F_2

i.e., C is $\left(\frac{-6+6}{2}, \frac{0+0}{2}\right) = (0, 0)$

The length of the transverse axis 2a = 8, => a = 4

$F^1F^2 = 2ae\ 12 \quad ae = 6$

$\therefore 4e = 6$

$e = \frac{6}{4} = \frac{3}{2}$

$b^2 = a^2 (e^2-1)$

$b^2 = 16\left(\frac{9}{4} - 1\right) = \frac{16\times5}{4} = 20$

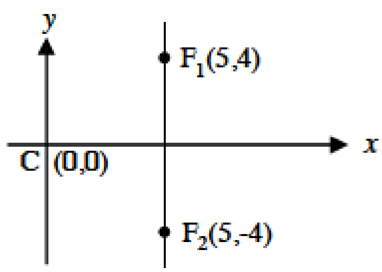
Thus the required of the equation is

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

42. Find the equation of the hyperbola whose foci are (5, $\pm$4) and Eccentricity is $\frac{3}{2}$

Solution:

From the given data the transverse axis is parallel to *y*-axis



And hence the equation of the hyperbola is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

The centre *C* (*h*, *k*) is the midpoint of *F*1 and *F*2

i.e., C is $\left(\frac{5+5}{2}, \frac{4-4}{2}\right) = (5, 0)$

$F_1F_2 = 2ae = \sqrt{(5-5)^2 + (4+4)^2} = 8$

$\therefore ae = 4$

$e = \frac{3}{2} \quad \Rightarrow a = \frac{8}{3}$

$b^2 = a^2 (e^2 - 1)$

$$b^2 = \frac{64}{9}\left(\frac{9}{4} - 1\right)$$

$$= \frac{80}{9}$$

Thus the required of the equation is

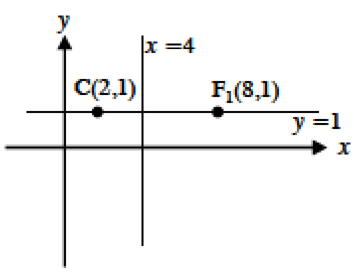
$$\frac{(y-0)^2}{64/9} - \frac{(x-5)^2}{80/9} = 1$$

43. Find the equation of the hyperbola whose centre is (2, 1) one of the foci is (8, 1) and the corresponding directrix is x = 4.

Solution:

From the given data equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



Centre C (h, k) is (2, 1)

$CF_1 = ae = 6$

The distance between the centre and directrix

$CZ = \frac{a}{e} = 2$

$ae \cdot \frac{a}{e} = 6 \times 2$ => $a^2 = 12$

$$\frac{ae}{a/e} = \frac{6}{2}$$ => $e^2 = 3$

$b^2 = a^2 (e^2-1)$

$b^2 = 12\,(3-1)$

$= 24$

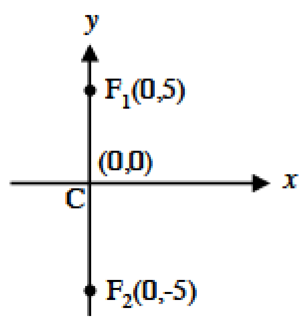
Thus the required of the equation is

$$\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$$

44. Find the equation of the hyperbola whose foci are $(0, \pm 5)$ and the length of the transverse axis is 6.

Solution:

From the given data the transverse axis is along *y*-axis and



hence the equation is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

The centre $C\,(h, k)$ is the midpoint of F_1 and F_2

i.e., C is $\left(\frac{0+0}{2}, \frac{5-5}{2}\right) = (0, 0)$

The length of the transverse axis 2a = 6, => a = 3

F_1F_2 = 2ae = 10 ae = 5

$e = \frac{5}{3}$

$b^2 = a^2 (e^2-1)$

$b^2 = 9\left(\frac{25}{9} - 1\right) = 16$

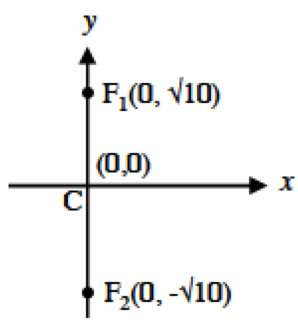
Thus the required of the equation is

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

45. Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ and passing through (2, 3).

Solution:

From the data, the transverse axis is along the *y*-axis.



∴ it is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Given that the foci are $(0, \pm ae) = (0, \pm\sqrt{10})$

$\Rightarrow ae = \sqrt{10}$

Also $b^2 = a^2 (e^2 - 1) = a^2e^2 - a^2$

$b^2 = 10 - a^2$

$\therefore$ Equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{10-a^2} = 1$

It passes through (2, 3),

$$\frac{9}{a^2} - \frac{4}{10-a^2} = 1$$

$$\frac{9(10-a^2)-4a^2}{a^2(10-a^2)} = 1$$

$$90 - 9a^2 - 4a^2 = 10a^2 - a^4$$

$$a^4 - 23a^2 + 90 = 0$$

$$(a^2 - 18)(a^2 - 5) = 0$$

$$a^2 = 18 \text{ or } 5$$

If $a^2 = 18$, $b^2 = 10 - 18 = -8$ which is impossible.

If $a^2 = 5$, $b^2 = 10 - 5 = 5$

$\therefore$ Equation of the hyperbola is

$$\frac{y^2}{5} - \frac{x^2}{5} = 1 \text{ or } \quad y^2 - x^2 = 5$$

46. Find the equations and length of transverse and conjugate axes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Solution:

The centre is at the origin, the transverse axis is along x-axis and the conjugate axis is along the y-axis.

i.e., transverse axis is x-axis i.e., $y = 0$ and

the conjugate axis y-axis i.e., $x = 0$.

Hence $a^2 = 9$, $b^2 = 4$

$\Rightarrow a = 3$, $b = 2$

$\therefore$ Length of transverse axis = $2a = 6$

Length of conjugate axis = $2b = 4$

47. Find the equations and length of transverse and conjugate axes of the hyperbola $16y^2 - 9x^2 = 144$

Solution:

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

The centre is at the origin,

The transverse axis is along y-axis, and

The conjugate axis is along x-axis.

$\therefore$ The transverse axis is y-axis, i.e. $x = 0$

The conjugate axis is x-axis i.e. $y = 0$.

Here $a^2 = 9$, $b^2 = 16$

$\Rightarrow a = 3$, $b = 4$

$\therefore$ The length of transverse axis = $2a = 6$

The length of conjugate axis = $2b = 8$

48. Find the equations and length of transverse and conjugate axes

of the hyperbola $9x^2 - 36x - 4y^2 - 16y + 56 = 0$

Solution:

$$9(x^2 - 4x) - 4(y^2 + 4y) = -56$$

$$9\{(x-2)^2 - 4\} - 4\{(y+2)^2 - 4\} = -56$$

$$9(x-2)^2 - 4(y+2)^2 = 36 - 16 - 56$$

$$9(x-2)^2 - 4(y+2)^2 = -36$$

$$4(y+2)^2 - 9(x-2)^2 = 36$$

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{4} = 1$$

$\frac{Y^2}{9} - \frac{X^2}{16} = 1$ Where $X = x - 2$; $Y = y + 2$

Clearly the transverse axis is along y-axis and

the conjugate axis is along x-axis.

i.e. transverse axis is y-axis or $X = 0$ i.e., $x - 2 = 0$

The conjugate axis is X-axis or $Y = 0$ i.e., $y + 2 = 0$

Here $a2 = 9$, $b2 = 4$

$\Rightarrow a = 3$, $b = 2$

The length of transverse axis = $2a = 6$

The length of conjugate axis = $2b = 4$

49. Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Solution:

The centre is at the origin and

the transverse axis is along x-axis.

The equations of the directrices are $x = \pm\frac{a}{e}$

The equations of the latus rectum are $x = \pm ae$

Length of the latus rectum = $\frac{2b^2}{a}$

Here $a^2 = 9$, $b^2 = 4$

$$e = \sqrt{1+\frac{b^2}{a^2}}$$

$$e = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$\therefore$The equations of the directrices are

$$x = \pm\frac{3}{\sqrt{13}/3} = \pm\frac{9}{\sqrt{13}}$$

The equation of the latus rectum are $x = \pm\sqrt{13}$

Length of the latus rectum is

$$\frac{2b^2}{a} = \frac{8}{3}$$

50. Find the equation of directrices, latus rectum and length of latus rectum of the hyperbola $16x^2$-$9x^2 = 144$

Solution:

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

Here $a^2 = 9$, $b^2 = 16$ $e = \frac{5}{3}$

The transverse axis is along the y-axis.

$\therefore$ The equations of the directrices are $y = \pm\frac{a}{e}$

i.e., $y = \pm\frac{9}{5}$

The equation of the latus rectum are $y = \pm ae$

i.e., $y = \pm 5$

Length of the latus rectum is

$$\frac{2b^2}{a} = \frac{32}{3}$$

51. Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $9x^2 - 36x - 4y^2 - 16y + 56 = 0$

Solution: By simplifying we get

$\frac{Y^2}{9} - \frac{X^2}{4} = 1$ where Y= $y + 2$; $X = x - 2$

Here $a^2 = 9$, $b^2 = 4$

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$e = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$$ae = \sqrt{13}\ , \frac{a}{e} = \frac{9}{\sqrt{13}}$$

The transverse axis is along Y- axis.

The equations of the directrices are

(i) Y = $\frac{9}{\sqrt{13}}$ => y+2 = $\frac{9}{\sqrt{13}}$ => y = $\frac{9}{\sqrt{13}} - 2$

(ii) Y = $-\frac{9}{\sqrt{13}}$ => y+2 = $\frac{-9}{\sqrt{13}}$ => y = $\frac{-9}{\sqrt{13}} - 2$

The equation the latus rectums are

(i) $Y = \sqrt{13}$ => $y+2 = \sqrt{13}$ => $y = \sqrt{13} - 2$

(ii) $Y = -\sqrt{13}$ => $y+2 = -\sqrt{13}$ => $y = -\sqrt{13} - 2$

Length of the latus rectum is

$$\frac{2b^2}{a} = \frac{8}{3}$$

52. The foci of a hyperbola coincide with the foci of the ellipse $x2$ $\frac{x^2}{25} + \frac{y^2}{9} = 1$ Determine the equation of the hyperbola if its eccentricity is 2.

Solution:

The equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$\Rightarrow a^2 = 25, \quad b^2 = 9,$

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

$\therefore ae = 4$

The foci of the ellipse are $(\pm ae, 0) = (\pm 4, 0)$

Given that the foci of the hyperbola coincide with the foci

of the ellipse, foci of the hyperbola are $(\pm ae, 0) = (\pm 4, 0)$

Given that the eccentricity of the hyperbola is 2

$a(2) = 4$

$\Rightarrow a = 2$

For a hyperbola $b^2 = a^2 (e^2 - 1)$

$= a^2e^2 - a^2$

$= 16 - 4$

$= 12$

$\therefore$ The equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

53. Find the equation of the locus of all points such that the differences of their distances from (4, 0) and (−4, 0) is always equal to 2.

Solution:

By the property, the locus is a hyperbola.

Take the fixed points as foci.

$\therefore F_1$ is (4, 0) and F_2 is (−4, 0)

Let $P(x, y)$ be a point on the hyperbola.

$F_2P \sim F_2P$= length of transverse axis = $2a = 2$

$\therefore a = 1$

Centre is the midpoint of F_1F_2 = (0, 0)

Hence from the given data the

Hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$F_1F_2 = 2ae = 8$

$ae = 4 \Rightarrow e = 4$

$$b^2 = a^2 (e^2 - 1)$$

$$= 1(16 - 1) = 15$$

Thus the equation is $\frac{x^2}{1} - \frac{y^2}{15} = 1$

54. Find the eccentricity, centre, foci and vertices of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and also trace the curve

Solution:

$a^2 = 4, \quad b^2 = 5$

$e = \sqrt{1 + \frac{b^2}{a^2}}$

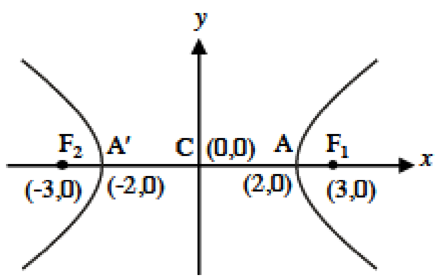
$e = \sqrt{1 + \frac{5}{4}} = \frac{3}{2}$

The transverse axis is along the x-axis

Centre : (0, 0)

Foci : $(\pm ae, 0) = (\pm 3, 0)$

Vertices : $(\pm a, 0) = (\pm 2, 0)$



55. Find the eccentricity, centre, foci and vertices of the hyperbola

$\frac{y^2}{6} - \frac{x^2}{18} = 1$ and also trace the curve.

Solution:

$a^2 = 6 \qquad b^2 = 18$

$e = \sqrt{1 + \frac{b^2}{a^2}}$

$e = \sqrt{1 + \frac{18}{6}} = 2$

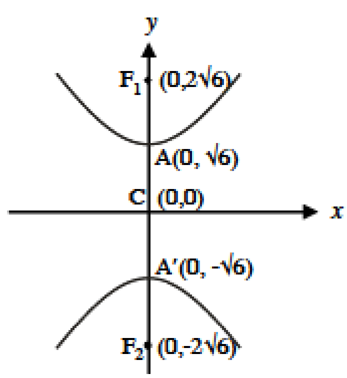
$\therefore ae = 2\sqrt{6}$

The transverse axis is along the *y*-axis

Centre : (0, 0)

Foci are : $(0, \pm ae) = (0, \pm 2\sqrt{6})$

Vertices : $(0, \pm a) = (0, \pm\sqrt{6})$



56. Find the eccentricity, centre, foci and vertices of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$ and also trace the curve.

Solution:

$9(x^2 - 2x) - 16(y^2 + 4y) = 199$

$$9\{(x-1)^2-1\} -16\{(y+2)^2-4\} = 199$$

$$9(x-1)^2-16(y+2)^2 = 199 + 9 - 64$$

$$9(x-1)^2-16(y+2)^2 = 144$$

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$$

i.e., $\frac{X^2}{16} + \frac{Y^2}{9} = 1$ where $X = x-1$

$Y = y + 2$

$a^2 = 16, \quad b^2 = 9$

$\Rightarrow e = \sqrt{1+\frac{b^2}{a^2}}$

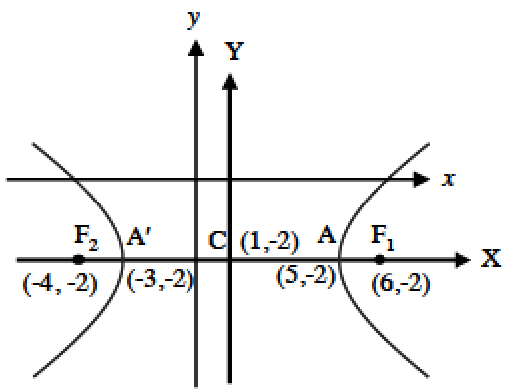
$e = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$

$ae = 4 \times \frac{5}{4} = 5$

The transverse axis is parallel to X-axis.

	Referred to X, Y	Referred to x, y X= x-1, Y= y+2
Centre	(0, 0)	X=0 ; Y=0 ⇨ X-1=0 ; y+2=0 ∴C (1, -2)
Foci	$(\pm ae, 0)$ is $(\pm 5, 0)$ (i) (5, 0)	(i) $X = 5$; $Y = 0$ $x-1 = 5$; $y + 2 = 0$

	(ii) $(-5, 0)$	$x = 6$; $y = -2$ $\therefore F_1 (6, -2)$ (ii) $X = -5$; $Y = 0$ $x - 1 = -5$; $y + 2 = 0$ $\therefore F_2 (-4, -2)$
Vertices	$(\pm a, 0)$ i.e. $(\pm 4, 0)$ (i) $(4, 0)$ (ii) $(-4, 0)$	(i) $X = 4$; $Y = 0$ $x - 1 = 4$; $y + 2 = 0$ $\therefore A (5, -2)$ ii) $X = -4$; $Y = 0$ $x - 1 = -4$; $y + 2 = 0$ $\therefore A' \ (-3, -2)$



57. Find the eccentricity, centre, foci and vertices of the following

hyperbola and draw the diagram : $9x^2-16y^2+36x+32y+164=0$

Solution:

$$9(x^2+4x)-16(y^2-2y)=-164$$

$$9\{(x+2)^2-4\}-16\{(y-1)^2-1\}=-164$$

$$9(x+2)^2-16(y-1)^2=-164+36-16$$

$$16(y-1)^2-9(x+2)^2=144$$

$$\frac{(y-1)^2}{9}-\frac{(x+2)^2}{16}=1$$

i.e., $\frac{Y^2}{9}+\frac{X^2}{16}=1$ where $X=x+2$

$Y=y-1$

$a^2=9,\quad b^2=16$

$\Rightarrow e=\sqrt{1+\frac{b^2}{a^2}}$

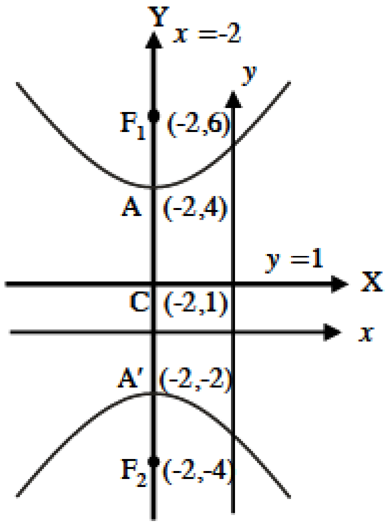
$e=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$

$ae=5$

The transverse axis is parallel to Y-axis.

	Referred to X, Y	Referred to x, y X= x+2, Y= y-1
Centre	(0, 0)	X=0 ; Y=0 ⇨ X+2=0 ; y-1=0

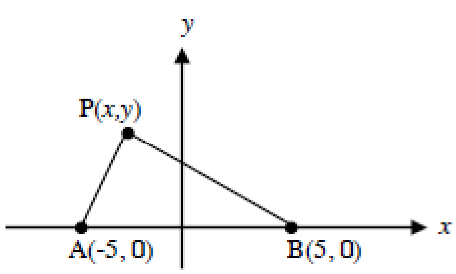
		∴C (-2, 1)
Vertices	(0, $\pm$a) i.e., (0, ± 3) (i) (0, 3) (ii) (0, -3)	(i) X= 0 ; Y=3 ⇨ X+2= 0 ; y-1=3 A (-2, 4) (ii) X = 0; Y = -3 X+2 = 0, y-1 =-3 X = -2, y = -2 A′ (-2, -2)
Foci	$(0, \pm ae)$ i.e., (0, ± 5) (i) (0, 5) (ii) (0, -5)	(i) X = 0 , Y = 5 X+2 =0, y-1 = 5 X = -2, y = 6 F_1 (-2, 6) (ii) X = 0, Y = -5 X+2 = 0; y-1= -5 x =-2; y = -4 F_2 (-2, -4)



58. Points *A* and *B* are 10 km apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to *A* than *B*. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Solution:

Given:



$PB - PA = 6$

$\sqrt{(x-5)^2+y^2} - \sqrt{(x+5)^2+y^2} = 6$

Simplifying we get $-9y^2 + 16x^2 = 14$

$$\frac{-y^2}{16} + \frac{x^2}{9} = 1$$

i.e. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

This is a hyperbola

59. Find the equations of the tangents to the parabola $y^2 = 5x$ from the point (5, 13). Also find the points of contact.

Solution:

The equation of the parabola is $y^2 = 5x$

Here $4a = 5$

$\Rightarrow a = \frac{5}{4}$

Let the equation of the tangent be $y = mx + \frac{a}{m}$

$$y = mx + \frac{5}{4m} \quad \text{----- (1)}$$

Since it passes through (5, 13) we have

$$13 = 5m + \frac{5}{4m}$$

$\therefore 20m^2 - 52m + 5 = 0$

$(10m - 1)(2m - 5) = 0$

$\therefore m = \frac{1}{10}$ or $m = \frac{5}{2}$

Using the values of m, we get the equations of tangents are

$2y = 5x + 1,$

$10y = x + 125.$

The points of contact are given by $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

where $a = \frac{5}{4}$ m = $\frac{5}{2}$, $\frac{1}{10}$

∴ the points of contact are $\left(\frac{1}{5}, 1\right)$, (125 , 25)

60. Find the equation of the tangent at $t = 1$ to the parabola $y^2 = 12x$

Solution:

Equation of the parabola is $y^2 = 12x$.

The equation of the tangent at 't' is $yt = x + at^2$

Here $4a = 12$

$\Rightarrow a = 3$ Also $t = 1$

∴ the equation of the tangent is $y = x + 3$

$$x - y + 3 = 0$$

61. Find the equation of the tangent and normal to the parabola

$x^2 + x - 2y + 2 = 0$ at (1, 2)

Solution:

The equation of the parabola is $x^2 + x - 2y + 2 = 0$

Equation of the tangent at (x_1, y_1) to the given parabola is

$$xx1 + \frac{x+x_1}{2} - 2\frac{(y+y_1)}{2} + 2 = 0$$

i.e., $x(1) + \frac{x+1}{2} - 2\frac{(y+2)}{2} + 2 = 0$

On simplification we get $3x - 2y + 1 = 0$

Equation of the normal is of the form $2x + 3y + k = 0$

This normal passes through (1, 2)

∴ $2 + 6 + k = 0$ ∴ $k = -8$

$\therefore$ Equation of the normal is $2x + 3y - 8 = 0$

62. Find the equations of the two tangents that can be drawn from the point (5, 2) to the ellipse $2x^2 + 7y^2 = 14$

Solution:

Equation of the ellipse is $2x^2 + 7y^2 = 14$

i.e. $\frac{x^2}{7} - \frac{y^2}{2} = 1$

Here $a^2 = 7$, $b^2 = 2$

Let the equation of the tangent be $y = m x + \sqrt{a^2m^2 + b^2}$

$\therefore y = m x + \sqrt{7m^2 + 2}$

Since this line passes through the point (5, 2) we get

$$2 = 5m + \sqrt{7m^2 + 2}$$

i.e. $2 - 5m = \sqrt{7m^2 + 2}$

$\therefore (2 - 5m)^2 = 7m^2 + 2$

$4 + 25m^2 - 20m = 7m^2 + 2$

$18m^2 - 20m + 2 = 0$

$9m^2 - 10m + 1 = 0$

$\therefore (9m - 1)(m - 1) = 0$

$\therefore m = 1$ or $m = \frac{1}{9}$

To find the equations of the tangents, use slope-point form

(i) $m = 1$,

The equation is $y - 2 = 1(x - 5)$ i.e., $x - y - 3 = 0$

(ii) $m = 1/9$

The equation is $y - 2 = \frac{1}{9}(x - 5)$, i.e., $x - 9y + 13 = 0$.

Thus the equations of the tangents are

$$x - y - 3 = 0,$$

$$x - 9y + 13 = 0$$

63.Find the equation of chord of contact of tangents from the point (2, 4) to the ellipse $2x^2 + 5y^2 = 20$

Solution:

The equation of chord of contact of tangents from (x_1, y_1) to $2x^2 + 5y^2 - 20 = 0$ is

$$2xx_1 + 5yy_1 - 20 = 0$$

$\therefore$ The required equation from (2, 4) is

$$2x(2) + 5y(4) - 20 = 0$$

$$\text{i.e. } x + 5y - 5 = 0$$

64. Find the separate equations of the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Solution:

The combined equation of the asymptotes differs from the hyperbola by a constant only.

$\therefore$ the combined equation of the asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$$

Consider $3x^2 - 5xy - 2y2 = 3x2 - 6xy + xy - 2y^2$

$$= 3x\,(x - 2y) + y(x - 2y)$$

` $$= (3x + y)\,(x - 2y)$$

$\therefore$ The separate equations are $3x + y + l = 0$,

$$x - 2y + m = 0$$

$$\therefore (3x + y + l)\,(x - 2y + m) = 3x^2 - 5xy - 2y^2 + 17x + y + k$$

Equating the coefficients of x, y terms and constant term,

We get $\quad l + 3m = 17$ ---- (1)

$-2l + m = 1$ ----- (2)

$$l\,m = k$$

Solving (1) and (2) we get $l = 2$, $m = 5$ and $k = 10$

Hence separate equations of asymptotes are

$$3x + y + 2 = 0,$$

$$x - 2y + 5 = 0$$

The combined equation of asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$$

65. Find the equation of the hyperbola which passes through the point (2, 3) and has the asymptotes $4x + 3y - 7 = 0$ and $x - 2y = 1$.

Solution:

The separate equations of the asymptotes are

$$4x + 3y - 7 = 0,$$

b $x - 2y - 1 = 0$

$\therefore$ combined equation of asymptotes is

$$(4x + 3y - 7)(x - 2y - 1) = 0$$

The equation of the hyperbola differs from this combined equation of asymptotes by a constant only.

$\therefore$ the equation of the hyperbola is of the form

$$(4x + 3y - 7)(x - 2y - 1) + k = 0$$

But this passes through (2, 3)

$$(8 + 9 - 7)(2 - 6 - 1) + k = 0 \ \therefore k = 50$$

$\therefore$ The equation of the corresponding hyperbola is

$$(4x + 3y - 7)(x - 2y - 1) + 50 = 0$$

i.e., $4x^2 - 5xy - 6y^2 - 11x + 11y + 57 = 0$

66 . Find the angle between the asymptotes of the hyperbola

$3x^2 - y^2 - 12x - 6y - 9 = 0$

Solution:

$$3x^2 - y^2 - 12x - 6y - 9 = 0$$

$$3(x^2 - 4x) - (y^2 + 6y) = 9$$

$$3\{(x-2)^2 - 4\} - \{(y+3)^2 - 9\} = 9$$

$$3(x-2)^2 - (y+3)^2 = 12$$

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{12} = 1$$

Here $a = 2,\ b = \sqrt{12} = 2\sqrt{3}$

The angle between the asymptotes is

$$2\alpha = 2\tan^{-1}\frac{b}{a}$$

$= 2\tan^{-1}\frac{2\sqrt{3}}{2}$

$= 2\tan^{-1}\sqrt{3}$

$= \frac{2\pi}{3}$

67. Find the angle between the asymptotes to the hyperbola

$3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Solution:

Combined equation of the asymptotes differs from that of they parabola by a constant only.

$\therefore$ Combined equation of asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$$

$3x^2 - 5xy - 2y^2 = 3x^2 - 6xy + xy - 2y^2$

$= 3x(x - 2y) + y(x - 2y)$

$= (x - 2y)(3x + y)$

$\therefore$ Separate equations are $x - 2y + l = 0$,

$$3x + y + m = 0$$

Let m_1 and m_2 be the slopes of these lines,

Then $m_1 = \frac{1}{2}$, $m_2 = -3$

$\therefore$ angle between the lines is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{1/2 - (-3)}{1 + 1/2(-3)}\right|$$

$= 7$

$\theta = \tan^{-1}(7)$

68. Prove that the product of perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is constant and the value is $\frac{a^2b^2}{a^2+b^2}$.

Solution:

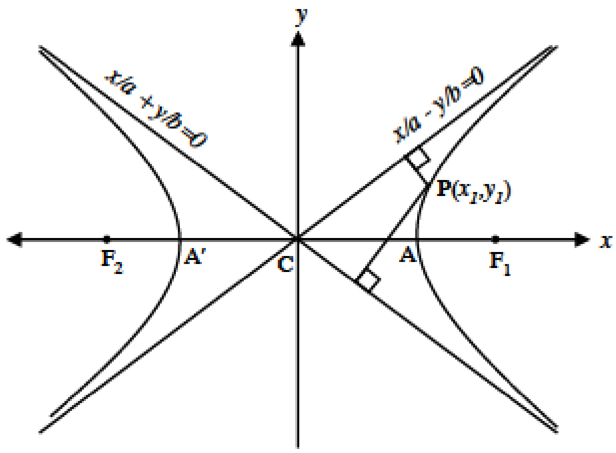
Let p(x_1, y_1) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$

The perpendicular distance from asymptote

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ is } \frac{\frac{x_1}{a} - \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \text{ and to}$$

$$\frac{x}{a} + \frac{y}{b} = 0 \text{ is } \frac{\frac{x_1}{a} + \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$



Product of perpendicular distances = $\dfrac{\frac{x_1}{a}+\frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \cdot \dfrac{\frac{x_1}{a}-\frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

$$= \frac{\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}}{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$= \frac{1}{\frac{b^2+a^2}{a^2b^2}}$$

$$= \frac{a^2b^2}{b^2+a^2}$$

This is constant.

69. Find the equation of the standard rectangular hyperbola whose centre is $\left(-2,\frac{-3}{2}\right)$and which passes through the point $\left(1,\frac{-2}{3}\right)$

Solution:

The equation of the standard rectangular hyperbola with centre at (h, k) is $(x - h)(y - k) = c^2$

The centre is $\left(-2, \frac{-3}{2}\right)$.

$\therefore$ the equation of the standard rectangular hyperbola is

$$(x+2)\ \left(y+\frac{3}{2}\right) = c^2$$

It passes through $\left(1, \frac{-2}{3}\right)$

$$(x+2)\left(\frac{-2}{3} + \frac{3}{2}\right) = c^2$$

$$= \frac{5}{2}$$

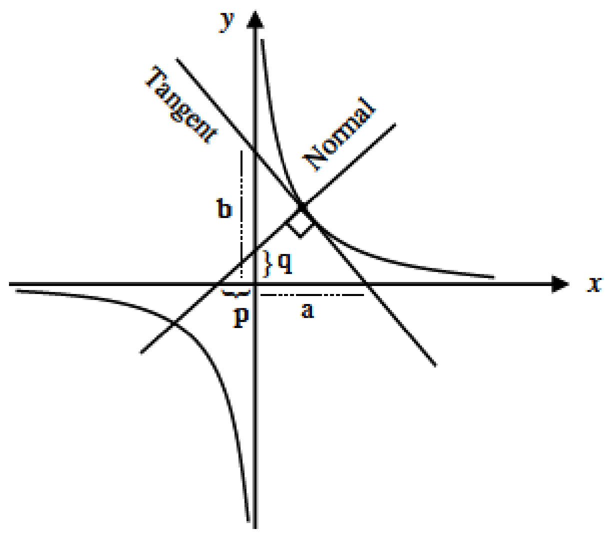
Hence the required equation is

$$(x + 2)\ \left(y+\frac{3}{2}\right) = \frac{5}{2}$$

or

$$2xy + 3x + 4y + 1 = 0$$

70. The tangent at any point of the rectangular hyperbola $xy = c^2$ makes intercepts a, b and the normal at the point makes intercepts p, q on the axes. Prove that $ap + bq = 0$



Solution:

Equation of tangent at any point 't' on $xy = c^2$ is $x + yt^2$

or

$$\frac{x}{2ct} + \frac{y}{2c/t} = 1$$

$\therefore$ $intercept\ on\ the\ axes\ are\ a = 2ct, b = \frac{2c}{t}.$

Equation of normal at 't' on $xy = c^2$ is $y - xt^2 = \frac{c}{t} - ct^3$

$$\frac{x}{\left(\frac{\frac{c}{t} - ct^3}{-t^2}\right)} + \frac{y}{\left(\frac{c}{t} - ct^3\right)} = 1$$

$\therefore$ $intercept\ on\ axes\ are\ p = \frac{-1}{t^2}\left(\frac{c}{t} - ct^3\right), q = \frac{c}{t} - ct^3$

$$\therefore ap + bq = 2ct\left(\frac{-1}{t^2}\right)\left(\frac{c}{t} - ct^3\right) + \frac{2c}{t}\left(\frac{c}{t} - ct^3\right)$$

$$= \frac{2c}{t}\left(\frac{c}{t} - ct^3\right) + \frac{2c}{t}\left(\frac{c}{t} - ct^3\right)$$

$$= 0$$

71. Show that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact.

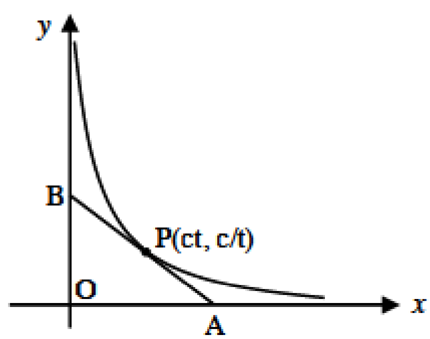
Solution:

The equation of tangent at P$\left(ct, \frac{c}{t}\right)$is $x + yt^2 = 2ct$

Putting $y = 0$ in this equation

we get the co-ordinates of A as$(2ct, 0)$.

Putting $x = 0$ we get the co-ordinates of B as$\left(0, \frac{2c}{t}\right)$



The mid-point of AB is $\left(\frac{2ct+0}{2}, \frac{0+\frac{2c}{t}}{2}\right) = \left(ct, \frac{c}{t}\right)$

This is the point P. This shows that the tangent is bisected at the point of contact.

4. Analytical Geometry

Exercise sums:

1. Find the equation of the parabola if

i) Focus : (2, -3) : directrix : 2y – 3 =0

ii) Focus : (-1, 3) : directrix : 2x + 3y =3

iii) Vertex : (0, 0) : focus : (0, -4)

iv) Vertex : (1, 4) : focus : (-2, 4)

v) Vertex : (1, 2) : latus rectum : y =5

vi) Vertex : (1, 4) : open leftward and passing through the point (-2, 10)

vii) Vertex : (3, -2): open downward and the length of the latus rectum is 8

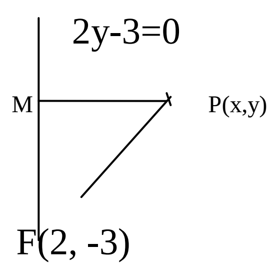
viii) Vertex : (3, -1): open rightward : the distance between the latus rectum and the directrix is 4.

ix) Vertex : (2, 3) : open upward ; and passing through the point (6, 4)

Solution : i) Focus : (2, -3) : directrix : 2y-3 =0

Let P(x, y) be any point on the parabola. If PM is drawn, perpendicular to the directrix

$$\frac{FP}{PM} = e=1$$



$\Rightarrow$ $FP^2 = PM^2$

$$(x+1)^2 + (y-3) = \left[\frac{(2x+3y-3)^2}{\sqrt{0^2+2^2}}\right]$$

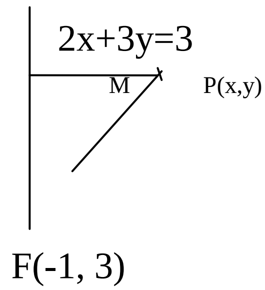
$\Rightarrow$ $4x^2-16x+36y+43=0$

(ii) Focus : (-1, 3) ; directrix : 2x+3y=3

Let P(x, y) be any point on the parabola. If PM is draw, perpendicular to the directrix

$\frac{FP}{PM}$ = e=1

$\Rightarrow$ $FP^2 = PM^2$

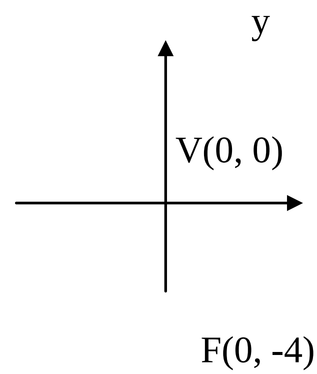


$(x+1)^2 + (y-3) = \left[\frac{(\pm(2x+3y-3)}{\sqrt{2^2+3^2}}\right]^2$

$\Rightarrow 9x^2-12xy+4y^2+38x-60y+121=0$

(iii) Vertex : (0,0) ; focus : (0, -4)

From the given data the parabola is open downward.



The required equation is

$(x-h)^2 = -4a\ (y-k)$

Here, the vertex (h, k) is (0,0) and the distance between vertex and the focus VF = a

$A= \sqrt{(0-0)^2 + (0+4)^2} = \sqrt{16} = 4$

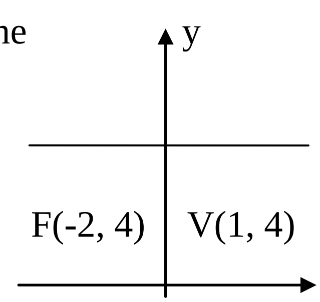
The required equation is

$(x - 0)^2 = -4(4)\ (y - 0)$

$x^2 = -16y$

iv) Vertex : (1, 4) ; focus : (-2, 4)

From the given data the parabola is open leftward. The Equation of the parabola is of the form



$(y - k)^2 = -4a(x - h)$

Here, the vertex (h, k) is (1, 4) and VF =a

$a = \sqrt{(1+2)^2 + (4-4)^2}$

a = 3

$(y - 4)^2 = -4(3)\ (x-1)$

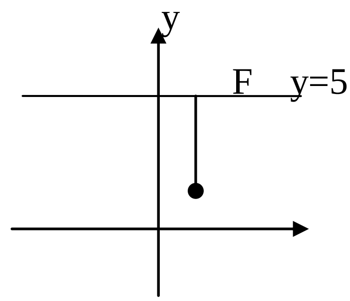
(y – 4)2 = -12(x-1)

v) Vertex : (1, 2) ; Equation of the latus rectum is y =5

From the given a data the parabola is open upward.

The equation is of the form

$$(x - h)^2 = 4a(y - k)$$



Here, the vertex V(h, k) is (1, 2)

LR

Draw a perpendicular from V to the latus rectum.

It meets at the focus. F is (1,5)

V(1,2)

Again VF = a =3

The required equation is

$$(x - 1)^2 = 4(3)(y-2)$$

$$(x-1)^2 = 12(y - 2)$$

vi) Vertex : (1, 4) ; open leftward and passing through the point (-2, 10)

Since it is open leftward, the equation of the parabola is of the form

$$(y - k) = -4a(x-h)$$

The vertex V(h, k) is (1,4)

$$(y-4)^2 = -4a(x-1)$$

But it passes through (-2, 10)

$$6^2 = -4a(-3) \Longrightarrow a = 3$$

vii) Vertex : (3, 2); open downward and the length of the latus rectum is8. Since it is open downward, the equation is of the form

$$(x - h)^2 = -4a(y-k)$$

The vertex V(h, k) is (3, -2)

The required equation is

$$(x-3)^2 = -4(2)(y+2)$$

$$(x - 3)^2 = -8(y+2)$$

viii) Vertex : (3, -1) ; open rightward ; the distance between the latus rectum and the directrix is 4.

Since it is open rightward, the equation is of the form,

$(y-k)^2 = 4a(x-h)$

The vertex (h, k) is (3, -1)

The distance between latus rectum and dirctrix = 2a =4

a =2

The required equation is $(y+1)^2 = (4(2)(x-3)$

$(y+1)^2 = 8(x-3)$

ix) Vertex : (2,3) ; open upward ; and passing through the point (6, 4)

Since it is open upward, the equation is of the form

$(x-h)^2 = 4a(x-3)$

The vertex V(h, k) is (z, 3)

$(x-2)^2 = 4a(y-k)$

But it passes through (6, 4)

$4^2 = 4a(y-3) \Longrightarrow a=4$

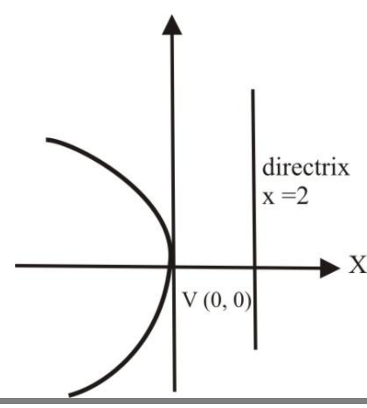
The required equation is $(x-2)^2 = 4(4)(y-3)$

Hence $(x-2)^2 = 16(y-3)$

2. Find the axis, vertex, focus, equation of directrix, latus rectum, length of the latus rectum for the following parabolas and hence sketch their graphs.

i) $y^2 = -8x$ (ii) $c^2 = 20y$ (iii) $(x-4)^2 = 4(y+2)$

iv) y2+8x-6y+1=0 (v) x2-6x-12y-3=0



Solution :

i) $y^2 = -8x$

$y^2 = -8x$

$(y-0)^2 = -4(2)(x-0)$

Hence (h, k) is (0, 0) and a =2

The parabola opens leftward.

Axis : The axis of symmetry is x axis.

y=0 is the axis.

Vertex : The vertex (h, k) id (0,0)

Focus : The focus F(-a, 0) is (-2, 0)

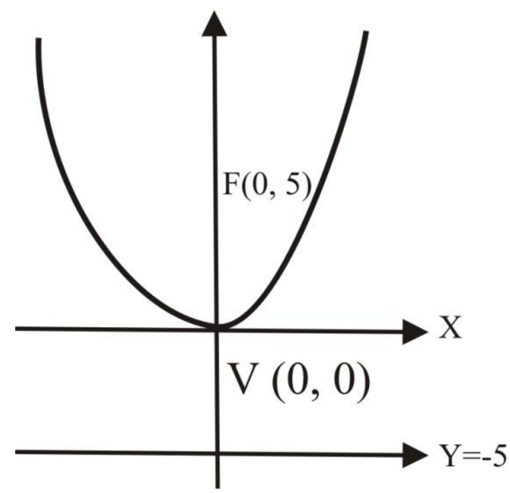
Directrix : The equation of the

x = -a ;   x=2. x-2=0

Equation of the latus rectum is

x = -a ;   x=-2. X+2=0

Length of the latus rectum 4a = 4(2) =8



ii) $x^2 = 20y$

$(x-0)^2 = 4(5)(y-0)$

Here (h, k) is (0, 0) and a =5

It is open upwards

Axis : y axis or x =0

Vertex : V(0, 0)

Focus : F(0, a) i.e., F(0, 5)

Directrix : y=-a i.e., y=5

i.e., y-5=0

Latus rectum : y=a i.e., y=5

i.e., y+5=0

: length =4a =20

(iii) $(x-4)^2 = 4(y+2)$

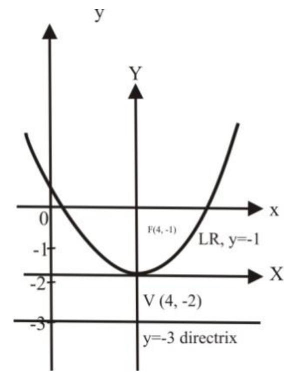
$X^2 = 4Y$ where X=x-4 ; Y=y+2

$X^2 = 4(1)y$ a=1

It is open upward

	referred to X, Y	**Referred to x, y X=x-4, Y=y+2**
Axis	y-axis i.e., X=0	X=0 ⇨ x-4=0
Vertex	(0, 0)	X=0 ; Y=0

		⇨ x-4=0 ; y+2=0 x=4 ; y=-2 V(4, -2)
Focus	(0, a) i.e., Y= -1	X = 0 ; Y=1 x-4=0 ; y+2=1 ⇨ x=4 ; y=-1 F(4, -1)
Directrix	Y=-a i.e., Y= -1	Y = -1⇨ y+2=-1 i.e., y +2=1 y+3 =0
Latus rectum	Y =a Y=1	Y=1⇨ y+2=1 i.e., y+1=0
	Length = 4a =4	



iv) y2+8x-6y+1=0

y26y=-8x-1

$(y-3)^2$-9=8x-1

$(y-3)^2$= 8x+8

$(y-3)^2$=8(x-1)

Where X = x-1

Y = y-3

Y^2= -4(2)x

⇨ a=2

The type is open leftward.

	referred to X, Y	**Referred to x, y X=x-1, Y=y-3**
Axis	X-axis i.e., Y=0	X=0 ⇨ y-3=0
Vertex	(0, 0)	X=0 ; Y=0 ⇨ x-1=0 ; y-3=0 x=1 ; y=3 V(1, 3)
Focus	(-a, 0) i.e., (-2,0)	X = -2 ; Y=0 x-2=-2 ; y-3=0 ⇨ x=-1 ; y=3 F(-1, 3)
Directrix	X=-a i.e., X= 2	X = 2 ⇨ x-1=-2 i.e., x +1=0 y-3 =0
Latus rectum	X =-a i.e.,X=-2	Y=-2 ⇨ x-1=-2 i.e., x+1=0
	Length = 4a =8	

v) $x^2-6x-12y-3=0$

$x^2-6x=12y+3$

$(x-3)^2 -9 =12y+3$

$(x-3)^2 =12y+12$

$(x-3)^2=12(y+1)$

$X^2= 12y$

Where X=x-3

Y=y+1

Here 4a =12

a=3

It is open upward.

	referred to X, Y	**Referred to x, y X=x-1, Y=y-3**
Axis	Y-axis i.e., X=0	X=0 ⇨ x-3=0
Vertex	(0, 0)	X=0 ; Y=0 x-3=0 ; y+3=0 ⇨ x=3 ; y=-1 V(3, -1)
Focus	(0, a) i.e., (0,3)	X = 0 ; Y=3 x-3=0 ; y+1=3 ⇨ x=3 ; y=2 F(3, 2)
Directrix	Y=-a i.e., Y= -3	Y = 2 ⇨ Y+1=-3 y+4 =0
Latus rectum	Y =a i.e., Y=3	Y=3 ⇨ y+1=3 i.e., y-2=0
	Length = 4a =12	4a=12

(3) If a parabolic reflector is 20cm in diameter and 5cm deep, find the distance of the focus from the centre of the reflector.

Solution :

Consider the parabolic reflector to be open rightwards.

i.e., $y^2=4ax$

In the fig., AB=20cm. VC=5cm

From the given data

(5, 10) is a point on it

$10^2 = 4a(5)$

⇨ 100=20a ⇨ a=5

The distane of the focus from the centre is 5

Focus (a,0) is (5, 0)

4. The focus of a parabolic mirror is at a distance of 8cm from its centre (vertex). If the mirror is 25cm deep, find the diameter of the mirror.

Solution :

Let AB be the diameter.

Consider the parabola to be open rightwards.

$y^2=4ax$

Focus is F(8, 0) a=8

The equation is $y^2=4(8)x$

$y^2=32x$

From the figure $(25,y_1)$ is a point on the parabola.

$y1^2=32(25)$

$y1^2=32x$

$y1=20\sqrt{2} \Rightarrow AC=20\sqrt{2}$

$AB=40\sqrt{2}$ is the diameter of the mirror.

5. A cable of a suspension bridge is in the form of a parabola whose span is 40mts. The road way is 5 mts below the lowest point of the cable. If an extra support is provided across the cable 30mts above the ground level, find the length of the support if the height of the pillars are 55 mts.

Solution :

Consider the suspension bridge to be open upwards. i.e., $x^2=4ay$

From the given data, the vertex of the bridge lies 5 meter above the rod away. The span of the bridge being 40 meters. The point A(20, 50) lies on the parabola.

$400 = 4a\ (50)$

a=2

The equation is $x^2=8y$

The point Q is $(x_1, 25)$ lies on the parabola

$$x1^2 = 8(25)$$

$$= 200$$

$$x_1 = 10\sqrt{2}$$

$PQ = 2x_1 = 20\sqrt{2}$ mts is the length of the support

Exercise 4.2

1. Find the equation of the ellipse if

(i) one of the foci is (0, -1) the corresponding directrix is

$3x+16=0$ and $e=\frac{1}{2}$

(ii) the foci are (2, -1), (0, -1) and $e=\frac{1}{2}$

(iii) the foci are $(\pm3,0)$ and the vertices are $(\pm5, 0)$

(iv) the centre is (3, -4) one of the foci is $(3+\sqrt{3}, -4)$ and $e=\frac{\sqrt{3}}{2}$

(v) the centre at the origin, the major axis is along x-axis, $e=\frac{2}{3}$ and passes through the point $\left(2, \frac{-5}{3}\right)$

(vi) the length of the semi major axis, and the latus rectum are 7 and $\frac{80}{7}$ respectively, the centre is (2,5) and the major axis is parallel to y-axis.

(vii) the centre is (3, -1) one of the foci is (6, -1) and passing through thepoint

(8, -1)

(viii) the foci are ($\pm$3,0) and the length of the latus rectum is $\frac{35}{5}$

(ix) the vertices are ($\pm$4, 0) and e= $\frac{\sqrt{3}}{2}$

Solution : (i) Let P(x,y) be a moving point. By definition

$\frac{FP}{PM}$ = e

$FP^2 = e^2 PM^2$

$(x-0)^2+(y+1)^2 = \left[\frac{3}{5}\right]^2 \left[\pm\frac{3x+16}{\sqrt{32}}\right]^2$

$25[x^2+(y+1)^2] = 9\left[\pm\frac{3x+16}{\sqrt{32}}\right]^2$

$25[x^2+y^2+2y+1] = (3x+16)^2$

$16x^2+25y^2-96x+50y-231=0$

(ii) The foci are (2, -1), (0-1) and e= $\frac{1}{2}$

F_1 (2, -1), F_2(0, -1)

The centre is the midpoint of F_1F_2

Centre C is $\left(\frac{2+0}{2}, \frac{-1-1}{2}\right)$

C is (1, -1)

$F_1 F_2 = 2ae=2$

$2a\frac{1}{2} = 2$

$\Rightarrow$ a=2

$b^2 = a^2(1-e^2)$

$= 4\left(1-\frac{1}{4}\right)$

$b^2 = 3$

From the given data the major axis is parallel to x-axis. The equation of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-1)^2}{3} = 1$

$\frac{(x-1)^2}{4} + \frac{(y-1)^2}{3} = 1$

(iii) The foci are (± 3, 0) and vertices are (± 5, 0)

The foci are F_1 (3, 0) and F_2 (-3, 0)

Vertices are A(5, 0) and A' (-5, 0)

The centre is the midpoint of AA'

C is $\frac{5-5}{2}, \frac{0+0}{2},$

From the given data the major axis is along x-axis.

The equation of the ellipse is $\frac{(x-h)^2}{a^2}, \frac{(y-k)^2}{b^2} = 1$, a > b

i.e., Here (h, k) is (0, 0)

Here CA = a =5 ; CF_1 = ae =3

$b^2 = a^2 (1-e^2)$ 5e = 3

$= 25\left(1- \frac{9}{25}\right)$ $e = \frac{3}{5}$

$b^2 = 16$

The equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(iv) The centre is (3, -4), one of the foci is (3+ $\sqrt{3}$, -4) and e= $\frac{\sqrt{3}}{2}$

Since the major axis is parallel to x-axis and the centre is at C(3, -4), the equation of the ellipse is

$$\frac{(x-3)^2}{a^2} + \frac{(y+4)^2}{b^2} = 1$$

CF = ae = $\sqrt{3}$,

$$a\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$\Rightarrow$ a = 2

$b^2 = a^2$

$= (1-e^2)$

$= 4\left(1 - \frac{3}{4}\right)$

$b^2 = 1$

The equation of the ellipse is $\frac{(x-3)}{4} + \frac{(y+4)^2}{1} = 1$

(v) Centre is at the origin, the major axis is along x-axis. $e = \frac{2}{3}$ and passes through the point $\left(2, \frac{-5}{3}\right)$

Since the major axis is along x-axis and the centre is at the origin, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It passes through the point $\left(2, \frac{-5}{3}\right)$

$$\frac{4}{a^2} + \frac{25}{9b^2} = 1$$

$e = \frac{2}{3}$; $b^2 = a^2(1-e^2)$

$b^2 = a^2\left(1 - \frac{4}{9}\right) = \frac{5}{9}a^2$

$9b^2 = 5a^2$

Solving (1) and (2)

$a^2 = 9$; $b^2 = 5$

The equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

(vi) The length of the semi major axis, and the latus rectum are 7 and $\frac{80}{7}$ respectively the centre is (2, 5) and the major axis is parallel to y-axis.

The major axis is parallel to the y-axis and the centre is (2, 5), the equation of the ellipse is

$$\frac{(x-2)^2}{b^2} + \frac{(y-5)^2}{a^2} = 1$$

From the given data a = 7 and

$$\frac{2b^2}{a} + \frac{80}{7} = 2b^2 = 80$$

$b^2 = 40$ and $a^2 = 49$

The equation of the ellipse is

$$\frac{(x-2)^2}{40} + \frac{(y-5)^2}{49} = 1$$

(vii) The centre is (3, -1) one of the foci is (6, -1) and passing through the point (8, -1) From the given data the major axis is parallel to the x-axis and the centre is (3,-1)

The equation of the ellipse is

$$\frac{(x-3)^2}{a^2} + \frac{(y+1)^2}{b^2} = 1$$

It passes through the point (8, -1)

$$\frac{(8-3)^2}{a^2} + \frac{(-1+1)^2}{b^2} = 1$$

$$\frac{25}{a^2} + 0 = 1$$

$a^2 = 25$ a =5

$CF_1 = ae = 3$

$5e = 3 \implies e = \frac{3}{5}$ $b^2 = a^2(1-e^2)$

$= 25\left(1 - \frac{9}{25}\right) = 16$

$\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1$ is the required equation of the ellipse.

(viii) The foci are $(\pm 3, 0)$, and the length of the latus rectum is $\frac{32}{5}$

$F_1(3,0)$, $F_2(-3, 0)$, $\frac{2b^2}{a} = \frac{32}{5}$

Centre is the midpoint of $F_1 F_2$

C is (0, 0)

From the given data the major axis is along x-axis.

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$F_1 F_2 = 2a\text{-}e = 6 \implies ae=3$

$a^2 e^2 = 9$

$b^2 = a^2(1 - e^2)$

$= a^2 - a^2 e^2$

$\implies a^2 - b^2 = 9$

It is given

$\frac{2b^2}{a} = \frac{32}{5}$

$b^2 = \frac{32a}{10}$

1. Becomes $a^2 - \frac{32a}{10} = 9$

$10a^2 - 32a - 90 = 0$

$(a - 5)(5a + 9) = 0$

$a = 5$ or $a = -\frac{9}{5}$

$a = \frac{9}{5}$ is not possible

$a = 5$

$b^2 = 16$

The equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

(ix) The vertices are $(\pm 4, 0)$ and $e = \frac{\sqrt{3}}{2}$

Vertex A is (4, 0) and A' is (-4, 0)

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$b^2 = a^2(1 - e^2)$

$= a^2\left(1 - \frac{3}{4}\right)$

$b^2 = \frac{a^2}{4}$

Also, $AA' = 2a = 8$

$a = 4$

$b^2 = 4$

The equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

2. If the centre of the ellipse is (4, -2) and one of the focus is (4, 2), find the other focus?

SOLUTION :

Centre C is (4, -2) and F_1 is (4, 2)

The midpoint of F_1 F_2 is the centre C

Let F_2 be (x_1, y_1)

$$\frac{x1+5}{2} = 4 \Rightarrow x_1 = 4$$

$$\frac{y1+2}{2} = 2 \Rightarrow y_1 = -6$$

F_2 is (4, -6)

3. Find the locus of a point which moves so that the sum of its distances from (3, 0) and (-3, 0) is 9

SOLUTION :

Take the fixed points (3, 0) and (-3, 0) as foci.

Let P(x, y) be the moving point.

From the known property,

F_1 P+F_2P = length of the major axis of the ellipse =9

i.e., 2a = 9 $\quad$ $a = \frac{9}{2}$

$F_1F_2 = 2ae = 6 \Rightarrow a\,e = 3$

$$e = \frac{1}{3}$$

Centre of the ellipse is the midpoint of F_1 F_2

C is (0, 0)

Also, $b^2 = a^2 (1 - e^2)$

$$b^2 = \frac{45}{4}$$

The equation of the ellipse is

$$\frac{x^2}{81/4} \qquad \frac{y^2}{45/4} = 1$$

4) Find the equations and length of major and minor axes of

i) $9x^2+25y^2 = 225$ (iii) $9x^2+4y^2 = 20$

ii) $5x^2 + 9y^2 + 10x - 36y - 4 = 0$ (iv) $16x^2+9y^2+32x-36y-92=0$

SOLUTION: (i) The equation of the ellipse is

$$9x^2+25y^2=225$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

The major axis is along x-axis and the minor axis is along y-axis.

The equation of major axis is y =0 and the equation of minor axis is x=0

$a^2 = 25$; $b^2 = 9$

$\Rightarrow$ $a = 5$; $b = 3$

The length of the major axis is 2a = 10

The length of the minor axis is 2b =6

ii) The equation of the ellipse is

$$5x^2+9y^2+10x-36y-4=0$$

$$(5x^2+10x) + (9y^2 - 36y) = 4$$

$5(x^2+2x) +9(y^2-4y) = 4$

$5 \{ (x +1)^2 -1\}+9\{(y-2)^2 -4\} =4$

$5(x+1)^2 +9(y-2)^2 -5-36=4$

$5(x+1)^2 + 9(y-2)^2 =45$

$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{5} =1$

Let X = x+1 and Y = y-2

$$\frac{X^2}{9} + \frac{Y^2}{5} =1$$

Hence the major axis is along the X-axis and the minor axis is along the Y-axis. The equation of major axis is Y =0

i.e., y -2=0

The equation of minor axis is X=0, i.e., x +1 =0

$a^2 =9$; $b^2 =5$ a=3, $b = \sqrt{5}$

The length of major axis is 2a =6

The length of minor axis is $2b =2\sqrt{5}$

iii) The equation of the ellipse

$9x^2+4y^2 = 20$

$\frac{x^2}{20/9} + \frac{y^2}{5} =1$

Here the major axis is along y-axis and the minor axis is along x-axis.

The equation of major axis is x=0 and the equation of minor axis is y =0

Her $a^2 =5$; $b^2 = \frac{20}{9}$

$\Longrightarrow$ $a=\sqrt{5}$; $b = \frac{2\sqrt{5}}{3}$

The length of the major axis is $2a = 2\sqrt{5}$

The length of the major axis is $2b = \frac{4\sqrt{5}}{3}$

iv) The equation of the ellipse is

$16x^2+9y^2+32x-36y-92 =0$

$(16x^2+32x)+(9y^2- 36y) =92$

$16(x^2+2x) +9(y^2-4y) =92$

$16\{(x+1)^2 - 1\} +9\{y-2)^2-4\}=92$

$16(x+1)^2+9(y-2)^2 =144$

$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{16} =1$

Let $x+1 =X$ and $y-2 =Y$

$\frac{X^2}{9} + \frac{Y^2}{16} = 1$

The major axis is along Y-axis

Its equation is $X=0$ i.e., $x+1=0$

The minor axis is along X-axis

The equation is $Y =0$ i.e., $y-2=0$

$a^2=16$, $b^2=9$ $a=4$, $b=3$

The length of major axis is $2a=8$

The length of minor axis is $2b=6$

5) Find the equations of directrices, latus rectum and lengths of latus rectums of the following ellipses:

i) $25x^2+169y^2=4225$ (ii) $9x^2+16y^2=144$

iii) $x^2+4y^2-8x-16y-68=0$ (iv) $3x^2+2y^2-30x-4y+23=0$

SOLUTION :

i) The equation of the ellipse is

$$25x^2+169y^2=4225$$
$$\frac{x^2}{169}+\frac{y^2}{25}=1$$

The major axis is along x-axis

$a^2=169,\ b^2=25$ $a=13,\ b=5$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{25}{169}}=\frac{12}{13}$$

The equation of directrices are

$$x=\pm\frac{a}{e}=\pm\frac{13}{12/13}=\pm\frac{169}{12}$$

$$\text{i.e., } x=\pm\frac{169}{12}$$

Equation of the latus rectum are $x=\pm ae$

$$x=\pm(13)\left(\frac{12}{13}\right)$$

$$x=\pm 12$$

Length of the latus rectum $=\frac{2b2}{a}=\frac{50}{13}$

ii) The equation of the ellipse is

$$9x^2+16y^2=144$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The major axis is along x-axis

$a^2 = 16, \quad b^2 = 9$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

The equation of directrices are $x = \pm \frac{a}{e}$

$$x = \pm \frac{4}{\frac{\sqrt{7}}{4}}$$

i.e., $x = \pm \frac{16}{\sqrt{7}}$

Equation of the latus rectum are $x = \pm$ a e

$$x = \pm\sqrt{7}$$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$

(iii) The equation of the ellipse is

$$x^2+4y^2-8x-16y-68=0$$

$$(x^2-8x) + (4y^2 - 16y) = 68$$

$$\{(x-4)^2 - 16\} + 4\{y^2-4y\} = 68$$

$$(x-4)^2 - 16+4\{(y-2)^2 - 4\} = 68$$

$$(x-4)^2+4(y-2)^2 = 100$$

$$\frac{(x-4)^2}{100} + \frac{(y-2)^2}{25} = 1$$

The major axis is parallel to x-axis.

Let X=x-4 and Y=y-2

$\frac{X^2}{100} + \frac{Y^2}{25} = 1$

$a^2 = 100$, $b^2 = 25$ a=10, b=5

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{25}{100}} = \sqrt{\frac{75}{100}}$

$= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

The equation of directrices are $X = \pm\frac{a}{e}$

$\Rightarrow X = \pm\frac{20}{\sqrt{3}}$

i.e., $x - 4 = \pm\frac{20}{\sqrt{3}}$

$x = 4 \pm\frac{20}{\sqrt{3}}$ are the diretrices

Equation of the latus rectum are $X = \pm ae$

$X = \pm 5\sqrt{3}$

i.e., $x - 4 = \pm 5\sqrt{3}$ $x = 4 \pm 5\sqrt{3}$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{2(25)}{10}$

$= 5$

iv) The equation of the ellipse is

$3x^2+2y^2-30x-4y+23=0$

$(3x^2-30x) + (2y^2-4y) = -23$

$3(x^2-10x) + 2(y^2-2y) = -23$

$3\{(x-5)^2 - 25\} + 2\{(y-1)^2 = -23$

$$3(x-5)^2 + 2(y-1)^2 = 54$$

The major axis is parallel to y-axis.

Let × = × -5 and Y = y -1

$\therefore X^2/18 + Y^2/27 = 1$ $a^2 = 27, b^2 = 18$, $e = 1 - b\sqrt{1 - \frac{b^2}{a^2}}$

$= \sqrt{1 - \frac{18}{27}} = \sqrt{\frac{9}{27}} = 1/\sqrt{3}$

$e = 1/\sqrt{3}$

$y = \pm 3\sqrt{3/1}/\sqrt{3} = \pm 9$

$y = 9 = y - 1 = 9 = y = 10$

$y = -9 = y - 1 = -9 = y = -8$

Thus $y = 10$ and $y = -8$ are the directrices.

The equation of the L.R are $Y = \pm ae$ $Y = \pm 3\sqrt{3}\ (1/\sqrt{3})$

i.e., $Y = \pm 3$ $Y = 3 \Rightarrow y - 1 = 3 = 4$

$Y = 3 \Rightarrow y-1 = -3 = -2$

Thus the quation of L.R. are $y = 4$ and $y = -2$

The length of the latus rectum $= 2b^2 = 2\times 18$

$= 4\sqrt{3}$

6. Find the eccentricity, centre, foci, vertices of the following ellipses and draw the diagram

(i) $16x^2+25y^2=400$ (ii) $x^2+4y^2-8x-16y-68=0$

(iii) $9x^2+4y^2=36$ (iv) $16x^2+9y^2+32x-36y=92$

SOLUTION :

(i) The equation of the ellipse is

$$16x^2+25y^2=400$$

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

Major axis is along x-axis

$$a^2=25,\ b^2=16,\quad e=\sqrt{1-\frac{b^2}{a^2}}$$

$$=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$$

$$e=\frac{3}{5}$$

Centre is (0, 0)

$$ae = 5x\,\frac{3}{5} = 3$$

Foci are ($\pm$ae, 0) i.e., ($\pm$3, 0)

Vertices are ($\pm$a 0, 0) i.e., ($\pm$ 5, 0)

ii) The equation of the ellipse is

$$x^2+4y^2-8x-16y-68=0$$

$$(x^2+8x) + (4y^2-16y) = 68$$

$$(x^2+8x) + 4(y^2-4y) = 68$$

$$\{(x-4)^2 -16\} +4\{(y-2)^2=68$$

$$(x-4)^2 + 4(y-2)^2 = 100$$

$$\frac{(x-4)^2}{100}+\frac{(y-2)^2}{25}=1$$

The major axis is parallel to x-axis

Let X = x-4 ; Y = y-2

$\frac{X^2}{100}+\frac{Y^2}{25}=1$

$a^2=100,\ b^2=25 \implies e=\sqrt{1-\frac{b^2}{a^2}}\sqrt{1-\frac{25}{100}}=\frac{\sqrt{3}}{2}$

$ae = 10 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

	referred to X, Y	**Referred to x, y** **X=x-4, Y=y-2**
Centre	(0, 0)	X=0 ; Y=0 x-4=0 ; y-2=0 $\Rightarrow$ x=4 ; y=2 C (4, 2)
Foci	$(\pm ae,0)$ i.e., $(\pm 5\sqrt{3}\ 0)$ (i) $(5\sqrt{3}, 0)$ (ii) $(-5\sqrt{3}, 0)$	$X=5\sqrt{3}$; Y=0 $x-4=5\sqrt{3}$; y-2=0 $x=4+5\sqrt{3}$; y=2 $F_1\ (4+5\sqrt{3}, 2)$ $X=-5\sqrt{3}$; Y=0 $x-4=-5\sqrt{3}$; y-2=0 $x=4-5\sqrt{3}$; y=2 $F_2\ (4-5\sqrt{3}, 2)$
Vertice	$(\pm a,0)$ i.e., $(\pm 10, 0)$ (i) (10,0) (ii) (-10, 0)	X = 10 ; Y=0 x-4=10 ; y-2=0 x=14 ; y=2 A (14, 2) X = -10 ; Y=0 x-4=-10 ; y-2=0 x=-6 ; y=2 A' (-6, 2)

(iii) The equation of the ellipse is

$9x^2+4y^2 = 36$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

The major axis is along y-axis

a^2=, b^2 =4

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{5}}{3}$$

$$ae = 3 \text{ x } \frac{\sqrt{5}}{3} = \sqrt{5}$$

Centre is (0, 0)

Foci are (0, $\pm$ae)

Vertices are (0, $\pm$a) i.e., (0, $\pm$3

iv) The equation of the ellipse is

$$16x^2+9y_2+32x-36y = 92$$

$$16(x^2+2x) +9(y^2-4y) = 92$$

$$16\{(x+1)^2-1\} +9\{(y-2)^2 =144$$

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{16} = 1$$

The major axis is parallel to y-axis.

Let X = x+1 ; Y = y-2

$$\frac{X^2}{9} + \frac{Y^2}{16} =1$$

$a^2 = 16$, b^2=9, e= $\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$; ae =4 x $\left(\frac{\sqrt{7}}{4}\right) = \sqrt{7}$

	referred to X, Y	**Referred to x, y** **X=x-4, Y=y-2**
Centre		X=0 ; Y=0

	(0, 0)	x+1=0 ; y-2=0 $\Rightarrow$ x= -1; y=2 Centre C (-1, 2)
Foci	$(0\pm ae, 0)$ i.e., $(0\pm 5\sqrt{7})$ (i) $(0+\sqrt{7})$ (ii) $(-5\sqrt{7}, 0)$ (iii) $(0, \sqrt{7})$	X=0; Y=0 X+1=0 ; y-2=$\sqrt{7}$ x= -1; y=2+$\sqrt{7}$ F_1 (-1,2+$\sqrt{7}$) X=0; Y= -$\sqrt{7}$ x+1=0 ; y=2=-$\sqrt{7}$ F_2 (-1, 2-$\sqrt{7}$)
Vertice	$(0\pm a,0)$ i.e., (0 ± 4) (i) (0, 4) (ii) (-0, -4)	X = 0; Y= 4 X+1=0 ; y-2=4 i.e., x=-1 ; y=6 A (-1, 6) X = 0 ; Y= -4 x+1=0 ; y-2=-4 x=-1 ; y=-2 A' (-1, -2)

7) A kho-kho player in a practice session while running realizes that the sum of the distances from the two tho-kho poles from him is alwys 8m. Find the equation of the path traced by him if the distance between the poles is 6m.

SOLUTION : From the given data, the two kho-kho poles be taken as the foci F_1 and F_2 Let P(x, y) be

$$F_1 P+F_2 P = 2a =8$$

$$a = 4$$

$$F_1F_2 = 2 ae =6$$

$$ae = 3$$

$$4e = 3$$

$\Longrightarrow$ e = $\frac{3}{4}$

Also $b^2 = a^2 (1-e^2) = 16\left(1 - \frac{9}{16}\right)$

$b^2 = 7$

The equation of the path is $\frac{x^2}{16} + \frac{y^2}{7} = 1$

8. A satellite is travelling around the earth in a elliptical orbit having the earth at a focus and of eccentricity ½. The shortest distance that the satellite gets to the earth is 400 kms. Find the longest distance that the satellite gets from the earth.

SOLUTION :

In the adjacent figure, earth is at F_1 The shortest

Distance of the satellite is F_1 A = 400 km. We have to find the longest distance of the satellite i.e., F_1A'

CA = a, CF_1 =ae, F_1A = 400 km

$F_1A = CA - CF_1$

= a – ae

400 = a $\left(1 - \frac{1}{2}\right)\left[e = \frac{1}{2}\right]$

a = 800 km

CA' = 800 and CF_1 = ae = 800 x $\frac{1}{2}$ = 400 km

$F_1A' = F_1C + CA'$

= 400 +800 = 1200 km

9) The orbit of the planet mercury around the sun is in elliptical shape with sun at a focus. The semi-major axis is of length 36 million miles and the eccentricity of the orbit is 0.206. Find (i) how close the mercury gets to sun? (ii) the greatest possible distance between mercury and sun.

SOLUTION :

In the adjacent figure, F_1 is the position of sun,

CA = 36 million miles, e = 0.206. The closest and farthest position of the planet mercury are A and A' respectively.

i) Closest distance F_1 A :

$F_1A = CA - CF_1$

$= a - ae = (a1-e) = 36\ (1-0.206)$

$= 36 \times 0.794$

Closet distance = 28.584 million miles

ii) The farthest position i.e., F_1A'

Let y_1 be the height of the arch from 9 ft right of the centre

$(9, y_1)$ is a point on the equation

$$\therefore \frac{9^2}{400} + \frac{y1^2}{256} = 1$$

$$\frac{y1^2}{256} = 1 - \frac{81}{400} = \frac{319}{400}$$

$$y_1^2 = 256\left(\frac{319}{400}\right)$$

$$y_1 = \frac{16\sqrt{319}}{20} = \frac{4}{5}\sqrt{319}\ \text{ft.}$$

$\therefore$ The height of the arch 9 ft from the right of the centre is $\frac{4}{5}\sqrt{319}$ ft.,

Exercise 4.3

1. Find the equation of the hyperbola if

(i) focus : (2, 3) ; corresponding directrix : x+2y = 5, e =2

(ii) centre : (0, 0) ; length of semi –transverse axis is 5 ; e = $\frac{7}{5}$ and the conjugate axis is along x-axis.

(iii) centre : (0, 0) ; length of semi-transverse axis is 6; e = 3, and the transverse axis is parallel to y-axis.

(iv) centre : (1, -2) ; length of the transverse axis is 8 ; e = $\frac{5}{4}$ and the transverse axis is parallel to x-axis.

(v) centre : (2, 5) ; the distance between the direct ices is 15, the distance between the foci is 20 and the transverse axis is 12

(vi) foci : (0, ≠ 8) ; length of transverse axis is 12

(vii) foci : (3, ≠5) ; e =3

(viii) centre : (1, 4) ; one of the foci (6,4) and the corresponding directrix is x = $\frac{9}{4}$

(ix) foci : (6, -1) and (-4, -1) and passing through the point (4, -1)

SOLUTION :

(i) Let P(x, y) be any point on the hyperbola. Draw PM perpendicular to the directrix.

By definition,$\frac{FP}{PM}$ = e

$\therefore$ $FP^2 = e^2$

$(x -2)^2 + (y – 3)^2 = e2\ PM2$

$(x-2)^2 + (y - 3)^2 = 2^{2}\left(\frac{x+2y-5}{\sqrt{1+4}}\right)^2$

$(x - 2)^2 + (y - 3)^2 = 4^{2}\frac{(x+2y-5)2}{5}$

$x^2-16xy – 11y^2 + 20x +50y – 35 =0$

(ii) Centre (0, 0) length of the semi –transverse axis is 5. e = $\frac{7}{5}$. The conjugate axis is along x-axis.

$\therefore$ The equation of the hyperbola is $\frac{y^2}{a^2}$ - $\frac{x^2}{b^2}$ = 1

As the transverse axis is parallel to y-axis.

Centre (0, 0) Length of the semi –transverse axis is = a =6, e =3

$b^2 = a^2 (e^2 -1)$

= 36 (9-1) = 288

$\frac{y^2}{36}$ - $\frac{x^2}{288}$ = 1

(iv) centre : (1, -2) ; length of the transverse axis is 8 ; e = $\frac{5}{4}$ and the transverse axis is parallel to x-axis.

$\therefore$ The equation of the hyperbola is $\frac{(x-h)^2}{a^2}$ - $\frac{(y+2)^2}{b^2}$ = 1

Centre (1, -2) length of the transverse axis 2a = 8 a = 4 and e = $\frac{5}{4}$

Here (h, k) is (1, -2)

The equation of the hyperbola is $\frac{(x-h)^2}{a^2}$ - $\frac{(y+2)^2}{b^2}$ = 1

a =4, e $\frac{5}{4}$, $b^2=a^2(e^2-1)$ =9

The equation of the hyperbola is $\frac{(x-1)^2}{16}$ - $\frac{(y+2)^2}{9}$ = 1

(v) Centre (2, 5), distance between the directrices = $\frac{2a}{e}$ = 15

$\therefore \frac{a}{e}$ = $\frac{15}{2}$ distance between foci = 2ae 20 $\therefore$ ae = 10

$\therefore \left(\frac{a}{e}\right)$ (ae) = $\frac{15}{2}$ x 10

a^2 = 75 also $\frac{ae}{a/e}\frac{10}{15/2} = \frac{4}{3}$ $e^2 = \frac{4}{3}$

b2 = a2 (e2 -1) $= 75\left(\frac{4}{3} - 1\right) = 25$

Since the transverse axis is parallel to y-axis, the equation of the hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x+h)^2}{b^2} = 1$$

Centre (h, k) is (2, 5) $\therefore \frac{(y-5)^2}{75} - \frac{(x-2)^2}{25} = 1$ is the required equation.

(vi) From the given data the transverse axis is alog the y-axis.

Also centre of the hyperbola is at the origin.

$\therefore$ The equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Length of the transverse axis = 2a =12

A =6

$F_1 F_2 = 16$

2ae = 16

2(6) e =16

$b^2 = a^2(e^2-1)$

$= 36\left(\frac{16}{9} - 1\right) \Longrightarrow b^2 = 28$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{36} - \frac{x^2}{28} = 1$

(vii) From the data, the transverse axis is parallel to the x-axis. Centre is the midpoint of F_1F_2

$\therefore$ C is $\left(\frac{3-3}{2}, \frac{5+5}{2}\right)$

$\therefore$ C is (0, 5)

$\therefore$ The equation of the hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Here (h, k) is (0, 5) $\therefore \frac{(x-0)^2}{a^2} - \frac{(y-5)^2}{b^2} = 1$

$F_1 F_2 = 2ae = 6 \quad \therefore ae = 3$ and $e = 3$

$\therefore 3a = 3 \implies a = 1$

$b^2 = a^2$ (e2-1)

= 1 (9-1)

$\therefore b^2 = 8$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{1} - \frac{(y-5)^2}{8} = 1$

(viii) From the data, the transverse axis is parallel to x-axis. $\therefore$ The equation of the hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Centre (h, k) is (1, 4)

$CF_1 = ae = 5$

The distance between the centre and the directrix CZ = $\frac{a}{e}$

$\therefore$ CZ = $\frac{a}{e} = \frac{9}{4}$-1 = $\frac{5}{4}$

$\therefore (ae)\left(\frac{a}{e}\right) = 5 \text{ x } \frac{5}{4} = \frac{25}{4}\ (4-1) = \frac{75}{4}$

The required equation is $\frac{(x-1)^2}{\frac{25}{4}} - \frac{(y+1)^2}{\frac{75}{4}}$

(ix) From the data, the transverse axis is parallel to the x-axis. ∴ The equation of the hyperbola is $\frac{(x-h)^2}{a^2}$ - $\frac{(y-k)^2}{b^2}$ =1

∴ C is $\left(\frac{-4+6}{2}, \frac{-1-1}{2}\right)$

∴ C is $(1, -1)$

∴ The equation of the hyperbola is $\frac{(x-h)^2}{a^2}$ - $\frac{(y-k)^2}{b^2}$ =1

F_1F_2 = 2ae = 10 ∴ a e=5

$b^2 = a^2 (e^2-1)$

$b^2 = a^2 e^2 - a^2$

∴ $b^2 = 25 - a^2$

∴The equation of the hyperbola is $\frac{(x-h)^2}{a^2}$ - $\frac{(y-k)^2}{b^2}$ =1

It passes through (4, -1)

∴$\frac{(4-1)^2}{a^2}$ - $\frac{(-1+1)^2}{25-a^2}$ =1 a^2 =9

(2) Find the equations and length of transverse and conjugate axes of the following hyperbolas

(i) $144x^2 - 25y^2 = 36000$ (ii) $8y^2$-2=16

(iii) $16x^2 - 9y^2$+96x+36y-36 =0

SOLUTION :

(i) $144x^2 - 25y^2 = 36000$

Equation of the hyperbola is $\frac{x^2}{25}$ - $\frac{y^2}{144}$ =1

$a^2 = 25 \Longrightarrow a=5$; $b2 = 144 \Longrightarrow b = 12$

The transverse axis is along x-axis.

$\therefore$ Equation of the transverse axis is y = 0

Equation of the transverse axis y =0

Length of the transverse axis = 2a = 2(5) =10

Length of the conjugate axis = 2b = 2(12) =24

(ii) $8y^2$-2=16

$\frac{y^2}{2} - \frac{x^2}{8} = 1$

The transverse axis is along y-axis.

Equation of the transverse axis is x = 0

Equation of the conjugate axis y =0

$a^2 = 2 \Longrightarrow a=\sqrt{2},\ b^2 = 8 \Longrightarrow b = \sqrt{8} = 2\sqrt{2}$

Length of the transverse axis (2a) = $2\sqrt{2}$

Length of the conjugate axis = (2b) = $4\sqrt{2}$)

(iii) $16x^2 - 9y^2 + 96x + 36y - 36 = 0$

$16(x^2+6x) - 9(y^2-4) = 36$

$16(x+3)^2 - 9(y^2-4) = 36 + 144 - 36 = 144$

$\frac{(x+3)^2}{9} - \frac{(y-2)^2}{16} = 1$

Put x+3 = X   y-2=Y

$\frac{X^2}{9} - \frac{Y^2}{16} = 1$

$a^2=9 \Longrightarrow a=3$

$b^2=16 \Longrightarrow b=4$

The transverse axis is along X-axis.

i.e., Equation of the transverse axis is Y=0 $\Longrightarrow$ a=3 = 0

Equation of the conjugate axis X=0 $\Longrightarrow$ = x+3=0

Length of the transverse axis =2a = 6

Length of the conjugate axis = 2b = 8

3) Find the equations of directrices, latus rectums and length of latus rectum of the following hyperbolas :

(i) $4x^2 - 9y^2 = 576$ (ii) $9x^2-4y^2-36+32y+8=0$

Solution :

(i) $4x^2 - 9y^2 = 576$

i.e., $\frac{x^2}{144} - \frac{y^2}{64} = 1$

$a^2 = 144 \Longrightarrow a=12,\ b^2=64 \Longrightarrow b=8$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{144+64}{144}} = \frac{\sqrt{13}}{3}$

Equation of the direct ices are $x = \pm\frac{a}{e}$ i.e., $x = \pm\frac{36}{\sqrt{13}}$

Equations of the latus rectum are

$x = \pm ae \Longrightarrow x = \pm\ 12 \times \frac{\sqrt{13}}{3}$ i.e., $x = \pm 4\sqrt{3}$

length of the latus rectum $= \frac{2b^2}{a} = \frac{2(64)}{12} = \frac{32}{3}$

(ii) $9x^2-4y^2-36+32y+8=0$

$9(x^2+4x) – 4(y^2-8y) = -8$

$9(x-2)^2 – 4(y^2-4)2 =-8+36 -64 =-36$

$4(y-4)^2 – 9(x-2)^2 =36$

$\frac{(y-4)^2}{9}$ - $\frac{(x-2)^2}{4}$ =1

Put y-4 = Y and x-2=X

$\frac{Y^2}{9}$ - $\frac{X^2}{4}$ = 1

The transverse axis is along Y-axis.

$a^2 = 9 \Longrightarrow a=3,\ b^2= 4 \Longrightarrow b=2$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{9+4}{9}} = \frac{\sqrt{13}}{9} = \frac{\sqrt{13}}{3}$

$a = 3\left(\frac{\sqrt{13}}{3}\right) = \sqrt{13}$

$\frac{a}{e} = \frac{3x3}{\sqrt{13}} = \frac{9}{\sqrt{13}}$

Equations of the latus rectums are

$\Longrightarrow$ Y -4 = $\pm\frac{9}{\sqrt{13}}$

$\Longrightarrow$ y = 4 = $\pm\sqrt{13}$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$

(4) Show that the locus of apont which moves so that the difference of its distances from the points (5, 0) and (-5, 0) is 8 $9x^2 - 16y^2 = 144$

SOLUTION :

By the property of hyperbola, the difference of the moving point from the two fixed points is the length of the transverse axis.

$\therefore 2ae = 10 \Rightarrow e = \frac{4}{4}$

$b^2=a^2 (e^2-1) = 16\left(\frac{25}{16} - 1\right) =9$

Centre is the midpoint of (5, 0) and (-5, 0). Centre is (0, 0)

Clearly the transverse axis is along x-axis.

$\therefore$ The type is $\frac{x^2}{a^2} - \frac{y^2}{b^2}= 1$ $\frac{x^2}{16} - \frac{y^2}{9} = 1$

i.e., $9x^2 - 16y^2 = 144$

5) Find the eccentricity, centre, foci and vertices of the following hyperbolas and draw their diagrams.

(i) $25x^2 - 16y^2 = 400$ (ii) $\frac{x^2}{9} - \frac{x^2}{25} =1$

(iii) $x^2-4y^2+6x+16y-11=0$

Solution :

(i) $25x^2 - 16y^2 = 400$ i.e., $\frac{x^2}{16} - \frac{x^2}{25} =1$

$a^2 =16 \Rightarrow a =4$, $b2 = 25 \Rightarrow b=5$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{16+25}{16}} = \frac{\sqrt{41}}{4} \Rightarrow ae = \frac{4\sqrt{41}}{4} = \sqrt{41}$

$\frac{a}{e} = \frac{4x4}{\sqrt{41}} = \frac{16}{\sqrt{41}}$

The transverse axis along x-axis

Centre : (0, 0)

Foci : ($\pm$ ae, 0) i.e., ($\pm\sqrt{41}$, 0)

Vertices : ($\pm$ a, 0) i.e., ($\pm$4, 0)

(ii) $\frac{x^2}{9} - \frac{x^2}{25} = 1$

$a^2 = 9 \Longrightarrow a = 3$

$b^2 = 25 \Longrightarrow b = 5$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{9+25}{9}} = \frac{\sqrt{34}}{3}$

$ae = \frac{3\sqrt{34}}{3} = \sqrt{34}$

$\frac{a}{e} = \frac{3\sqrt{34}}{3} = \sqrt{34}$

The transverse axis along y-axis

Centre : (0, 0)

Foci : (0, $\pm$ ae, 0) i.e., (0, $\pm\sqrt{34}$)

Vertices : (0, $\pm$ a) i.e., (0, $\pm$3)

(iii) $x^2-4y^2+6x+16y-11=0$

$(x^2+6x) - 4(y^2-4y) = 11$

$\{(x+3)^2 - 9\} - 4\{(y-2)^2-4\} = 11$

$(x+3)^2 - 4(y-2)^2 = 4$

$\frac{(x+3)^2}{4}$ - $\frac{(y-2)^2}{1}$ =1

$\frac{X^2}{4}$ - $\frac{Y^2}{1}$ =1

Where X = x+3, Y = y-2

$a^2=4 \Rightarrow a=2$

$b^2=1 \Rightarrow b=1$

$e = \sqrt{1+\frac{b^2}{a^2}} = \frac{\sqrt{5}}{2}$

$ae = 2.\frac{\sqrt{5}}{2} = \sqrt{5}$

$\frac{a}{e} = \frac{2x2}{\sqrt{5}} = \frac{4}{\sqrt{5}}$

	referred to X, Y	**Referred to x, y** **X=x+3, Y=y-2**
Centre	(0, 0)	X=0 ; Y=0 x+3=0 ; y-2=0 $\Rightarrow$ x=-3; y=2 $\therefore$ Centre C (-3, 2)
Foci	($\pm$ae,0) i.e., ($\pm\sqrt{5}$, 0) (i) ($\sqrt{5}$, 0) (ii) (-$\sqrt{5}$, 0)	X=$\sqrt{5}$; Y=0 x+3=$\sqrt{5}$; y-2=0 x=-3+$\sqrt{5}$; y=2 F_1 (-3+$\sqrt{5}$, 2) X=-$\sqrt{5}$; Y=0 x+3=-$\sqrt{5}$; y-2=0 x=-3-$\sqrt{5}$; y= 2 F_2 (-3-$\sqrt{5}$, 2)
Vertice	($\pm$ a, 0) i.e., ($\pm$ 2, 0) (i) (2,0) (ii) (-2, 0)	(i) X = 2 ; Y=0 X+3=2 ; y-2=0 i.e., x=-1 ; y=2 $\therefore$ A (-1, 2)

		X = -2 ; Y=0 X+3=-2 ; y-2=0 x=-5 ; y=2 A' (-5, 2)

(iv) $x^2-3y^2+6x+16y-18=0$

$(x^2+ 6x) – 3(y^2-2y) = 18$

$\{(x+3)^2 -9\}-3 \{(y-1)^2-4\} = -18$

$(x+ 1)^2 -3(y - 1)^2 = 12$

$3(y- 1)^2 -(x + 3)^2 = 12$

$\frac{(y+1)^2}{4} - \frac{(x+3)^2}{12} =1$

$\frac{Y^2}{4} - \frac{X^2}{12} =1$

Where X = x+3, Y = y-1

$a^2=4 \Rightarrow a =2$

$b^2=12 \Rightarrow b =2\sqrt{3}$

e=2 ae =2(2) =4, $\frac{a}{e}$=1. The transverse axis is parallel to y-axis.

	referred to X, Y	**Referred to x, y** **X=x+3, Y=y-2**
Centre	(0, 0)	X=0 ; Y=0 x+3=0 ; y-1=0 $\Rightarrow$ x=-3; y=1 $\therefore$ Centre C (-3, 1)
Foci	$(\pm ae,0)$ i.e., $(0, \pm 4)$ (i) (0, 4)	X=0; Y=4 x+3= 0 ; y-1=4 x=-3 ; y= 5 F_1 (-3, 5)

	(ii) (0, -4)	X=0; Y= -4 x+3=0; y-1=-4 x=-3- ; y= -3 F_2 (-3, -3)
Vertice	(0, $\pm$ a) i.e., (0,$\pm$ 2) (i) (0, 2) (ii) (0, -2)	X = 0 ; Y= 2 x+3=0 ; y-1=2 i.e., x=-3 ; y= 3 A (-3, 3) X = 0 ; Y= -2 X+3=0 ; y-1=-2 x=-3 ; y=-1 A' (-3, -1)

EXERCISE 4.4

1. Find the equations of the tangent and normal to the parabolas

(i) y^2 =12x at (3, -6)

(ii) x^2=9y at (-3, 1)

(iii) x^2+2x-4y+4=0 at (0, 1)

(iv) to the ellipse $2x^2$+3y2 = 6 at ($\sqrt{3}$, 0)

(v) to the hyperbola $9x^2 - 5y^2$ =31 at (2, -1)

SOLUTION :

(i) y^2 =12x at (3, -6)

Equation of the tangent a (x_1 y_1) to given parabola is yy1 = 6(x+x_1)

Equation of the tangent at (3, -6) is -6y =6 (x+x_1)

$-6y = 6(x+3)$

$6x+6y+18=0$

$x+y+3=0$

The equation of the normal is of the form $x - y + k = 0$

It passes through (3, -6)

$3+6+k=-9$

Equation of the tangent at (-3, 1) is

$$x(-3) = \frac{9}{2}(y+1)$$

$-6x=9y+9$

$6x+9y+9=0$

$2x+3y+3=0$

Equation of the normal is of the form $3x-2y+k=0$

It passes through (-3, 1)

$-9-2+k=0 \Rightarrow k=11$

$\therefore$ Equation of the normal is $3x-2y+11=0$

(iii) $X^2+2x-4+4=0$ at (0,1)

Equation of the tangent at $(x_1\ y_1)$ to the given parabola is

$xx_1 + (x+x_1) - 2(y+y_1) + 4 = 0$

Here (x_1, y_1) is (0,1)

$\therefore x(0) + (x+0) - 2(y+1) + 4 = 0$

i.e., $x-2y+2=0$

This is the equation of the tangent.

Equation of the normal is of the form 2x+y+k=0

It passes through (0,1)

$\therefore 2(0)+1+k=-1$

$\therefore$ Equation of the normal is 2x+y-1=0

(iv) $2x^2+3y^2=6$ at $(\sqrt{3},0)$

Equation of the tangent at (x_1,y_1) is

$$2xx_1+3yy_1=6$$

But (x_1,y_1) is $(\sqrt{3},0)$

$\therefore\ 2x(\sqrt{3})+3y(0)=6$

$$2\sqrt{3}x=6 \Rightarrow x=\sqrt{3}$$

$X=\sqrt{3}$ is the equation of the tangent.

Equation of the normal is of the form y = k

It passes through $(\sqrt{3},0)\ \therefore k=0$

Hence , the equation of the normal is y = 0

(v) To the hyperbola $9x^2-5y^2=31$ at (2,-1)

Equation of the tangent at (x_1,y_1) to the given hyperbola is

$9x^2-5y^2=31$

Here (x_1,y_1) is (2,-1)

$\therefore 9x(2)-5y(-1)=31$

$$18x+5y-31=0$$

Is the equation of the tangent .

Equation of the normal is the form 5x – 18y+k = 0

It passes through (2,-1)

$5(2) - 18(-1) + k = -28$

$\therefore$ Equation of the normal is $5x - 18y - 28 = 0$

2) Find the equation of the tangent and normal

(i) to the parabola $y^2 = 8x$ at $t = \frac{1}{2}$

(ii) to the ellipse $x^2+4y=32$ at $\theta = \frac{\pi}{4}$

(iii) to the ellipse 16x2+25y2=400 at $t = \frac{1}{\sqrt{3}}$

(iv) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{12} = 1$ at $\theta = \frac{\pi}{6}$

Solutin :

(i) to the parabola $y^2 = 8x$ at $t = \frac{1}{2}$

Equations of the tangent at ‘t’ to the parabola is

$yt = x+at^2$

At $t = \frac{1}{2}$ the equation is $y\left(\frac{1}{2}\right) = x+2\left(\frac{1}{2}\right)^2$

$\frac{y}{2} = x+\frac{1}{2}$

$y = 2x+1$

Equation of the normal at t is

$y+tx = 2at + at^3$

$y+\frac{1}{2}x = 2(2)\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^3$

$$y+\frac{x}{2}x = 2+\frac{1}{4}$$

$$4y+2x=8+1$$

$$2x+4y-9=0$$

(ii) to the ellipse $x^2+4y=32$ at $\theta = \frac{\pi}{4}$

Equation of the ellipse is $\frac{x^2}{32} + \frac{y^2}{8} = 1$

$$a^2 = 32 \implies a=4\sqrt{2}$$

$$b^2 = 8 \implies b=2\sqrt{2}$$

'θ' represents the point (a cos θ, b sin θ)

$$\implies \left(4\sqrt{2}con\ \frac{\pi}{4}, 2\sqrt{2}sin\frac{\pi}{4}\right)$$

$$\implies \left(4\sqrt{2}\left(\frac{1}{\sqrt{2}}\right),\ 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= (4, 2)$$

Equation of the tangent at $(x_1\ y_1)$ to the given ellipse is

$$xx_1 + 4yy_1=32$$

Here (x_1, y_1) is (4, 2)

$$4x +8y =32$$

i.e., x+2y -8=0 is the equation of the tangent.

Equation of the normal is of the form 2x-y+k=0

It passes through (4, 2)

$$2(4)-2+k=0 \implies k=-6$$

$\therefore$ Equation of the normal is 2x-y-6 =0

(iii) to the ellipse 16x2+25y2=400 at $t = \frac{1}{\sqrt{3}}$

Equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$a^2 = 25 \quad a \Rightarrow$

$b^2 = 16 \quad b \Rightarrow$

't' represents the point $\left(\frac{a(1-t^2)}{1+t^2}, \frac{b2t}{1+t^2}\right)$

It $t = \frac{1}{\sqrt{3}}$ then the point is $\left(\frac{5\left(1-\frac{1}{3}\right)}{1+\frac{1}{3}}, \frac{2(4)\left(\frac{1}{\sqrt{3}}\right)}{1+\frac{1}{3}}\right)$

i.e., $\left(\frac{5}{2}, 2\sqrt{3}\right)$

Equation of the tangent to the ellipse is

$$16x_1 + 25yy_1 = 400$$

Here (x_1, y_1) is $\left(\frac{5}{2}, 2\sqrt{3}\right)$

$\therefore 16x\left(\frac{5}{2}\right) + 25y\ (2\sqrt{3}) = 400$

$$40x + 50\sqrt{3}\ y = 400$$

$4x + 5\sqrt{3}\ y = 40$ is the equation of the tangent.

Equation of the normal is $5\sqrt{3}\ x - 4y + k = 0$

It passes through $\left(\frac{5}{2}, 2\sqrt{3}\right)$

$5\sqrt{3}\left(\frac{5}{2}\right)$-4 $2\sqrt{3}$ +k =0

$k + \frac{25\sqrt{3}}{2} - 8\sqrt{3} = 0$

$k + \frac{9\sqrt{3}}{2} = 0 \Rightarrow k = \frac{-9\sqrt{3}}{2}$

$\therefore$ Equation of the normal is

$5\sqrt{3}x - 4y\ \frac{-9\sqrt{3}}{2} = 0 \Rightarrow 10\sqrt{3x} - 8y - 9\sqrt{3} = 0$

(iv) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{12} = 1$ at $\theta = \frac{\pi}{6}$

$a^2 = 9 \Rightarrow a = 3$

$b^2 = 12 \Rightarrow b = 2\sqrt{3}$

'θ' represents the point (a sec θ, b tan θ)

i.e., $\left(3zec\frac{\pi}{6}, 2\sqrt{3}tan\frac{\pi}{6}\right)$ i.e, $\left(\frac{6}{\sqrt{3}}, 2\right)$ = $(2\sqrt{3}, 2)$

Equation of the tangent to the given hyperbola is $\frac{xx_1}{9} - \frac{yy_1}{12} = 1$

Here (x_1, y_1) is $(2\sqrt{3}, 2)$

$\frac{x,2\sqrt{3}}{9} - \frac{y2}{12} = 1$

$4\sqrt{x} - 3y = 18$

The equation of the normal is of the form $3x + 4\sqrt{3y} + k = 0$

It passes through $(2\sqrt{3})$

$\therefore 3(2\sqrt{3}) + 4(\sqrt{3}(2) + k = 0$

$k = -14\sqrt{3}$

$\therefore$ Equation of the normal is $3x+4\sqrt{3}=0$

(3) Find the equations of the tangents

(i) to the parabola $y^2 = 6x$, parallel to $3x - 2y+5 = 0$

(ii) to the parabola $y^2 = 6x$, perpendicular to the line $3x-y+8=0$

(iii) to the ellipse $\frac{x^2}{20} + \frac{y^2}{5} = 1$ which are perpendicular to $x+y+2=0$

(iv) to the hyperbola $4x^2-y^2=64$, which are parallel to

$10x - 3y + 9 = 0$

Solution :

(i) Find the equations of the tangents to the parabola $y^2 = 6x$, parallel to

$3x - 2y + 5 = 0$

The tangent is parallel to $3x - 2y + 5 = 0$

i.e., $y = \frac{3}{2}x + 5$

$\therefore$ The slope of the tangent $m = \frac{dx}{2dx}$ c

$y^2 = 6x$

Differentiating with respect to x

$2yy = 6 \quad 2y\left(\frac{3}{2}\right) = 6 \Rightarrow y = 2$

$4 = 6x \Rightarrow x = \frac{2}{3}$

$\therefore$ The point is $\left(\frac{2}{3}, 2\right)$

The tangent equation is $(y-y_1) = m(x-x_1)$

$$(y-2) = \frac{2}{3}\left(x\frac{2}{3}\right)$$

$$2y - 4 = 3x - 2$$

$$3x - 2y + 2 = 0$$

Alternative method :

Equation of any tangent is of the form $y = mx + \frac{a}{m}$

Here $m = \frac{3}{2}$; $a = \frac{3}{2}$

$\therefore y = \frac{3}{2}x + 1 \implies 3x-2y+2=0$

Caution : This form is applicable only when the vertex is (0, 0)

(ii) To the parabola y2 = 16x, perpendicular to the line 3x-y+8=0

$\therefore$ The slope of the normal $\left(-\frac{1}{m}\right)$ is 3

The slope of the tangent $m = -\frac{1}{3}$

Equation of any tangent is $y = mx + \frac{a}{m}$

i.e., $y = mx + \frac{a}{m}\frac{4}{\left(-\frac{1}{3}\right)}$ [Here a = 4

$y = -\frac{1}{3}x - 12 \implies x+3y+36=0$

(ii) To the ellipse $\frac{x^2}{20} + \frac{y^2}{20} = 1$, which are perpendicular to x+y+2=0

Here $a^2 = 20$, $b^2 = 5$

The tangent is perpendicular to x + y+2 =0 : i.e., y = -x -2

$\therefore$ Slope of the normal is $\left(-\frac{1}{m}\right)$ - 1

$\therefore$ Slope of the tangent is (m) = 1

Equation of any tangent is of the form

$$y = mx \pm \sqrt{a^2 + m^2 + b^2}$$

$$y = \left(\frac{10x}{3}\right) x \pm \sqrt{16\left(\frac{100}{9}\right) - 64}$$

$$= \frac{10x}{3} \pm \sqrt{\frac{1600 - 576}{3}}$$

$$10x-3y \pm 32 = 0$$

(4) Find the equation of the two tangents that can be drawn

(i) from the point (2, -3) to the parabola y^2 =4

(ii) from the point (1, 3) to the ellipse $4x^2+9y^2=36$

(iii) from the point (1, -1) to the hyperbola $2x^2-3y^2=6$

SOLUTION :

(i) Find the equation of the two tangents that can be drawn from the point (2, -3) to the parabola $y^2=4x$

Equation of any tangent is of the form $y = mx + \frac{a}{m}$

It passes through the point (2, -3)

$\therefore m(-3) = m^2(2) + 1$ [$\therefore$ a=1]

$2m^2+3m+1 = 0$

$(2m+1)(m+1) = 0$

$m = \frac{-1}{2}$; $m = -1$

$m = -1 \Rightarrow x+y+1=0$

$m = -3$; $m=1$

Equations of the tangents are

$(y - y_1) = m(x - x_1)$

i.e., $(y - 2) = m(x - 1)$

When $m = -3$

(1) $(y - 2) = 1(x - 1)$

$x - y + 1 = 0$

5) Prove that the line $5x + 12y = 9$ touches the hyperbola $x^2 - 9y^2 = 9$ and find its point of contact.

Solution : The condition for $y = mx + c$ to be a tangent to a hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c2 = a^2 m^2 - b^2$

$5x + 12y = 9 \Rightarrow y = \frac{-5}{12}x + \frac{3}{4}$ $m = \frac{-5}{12}, c = \frac{3}{4}$

$x^2 - 12y = 9 \Rightarrow y = \frac{x^2}{9} - \frac{y^2}{1} = 1 \Rightarrow a^2 = 9, b^2 = 1$

$c^2 = \frac{9}{16}$; $a^2 m^2 - b^2 = 9\left(\frac{25}{144}\right) - 1 = \frac{25}{144} - 1 = \frac{81}{144} = \frac{9}{16}$

$\Rightarrow c^2 = a^2 m^2 - b^2$

Thus the line $5x+12y = 9$ is a tangent to the hyperbola.

i.e., it touches the hyperbola.

The point of contact is $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$

$\frac{-a^2m}{c}$=-9 x $\left(\frac{-5}{12}\right)$ x $\frac{4}{3}$ = 5

$\frac{-b^2}{c}$= -1 x $\frac{4}{3}$ = $\frac{-4}{3}$

The point of contact is $\left(5, \frac{-4}{3}\right)$

(6) Show that the line x-y+4 =0 is a tangent to the ellipse $x^2+3y^2 = 12$. Find the co-ordinates of the point of contact.

Solution :

The condition for y = mx+c to be a tangent of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$c^2 = a^2m^2 + b^2$

x-y+4=0 $\Rightarrow$ y = x+4 $\Rightarrow$ m=1, c=4

$x^2+3y^2=12 \Rightarrow \frac{x^2}{12} + \frac{y^2}{4} = 1 \Rightarrow a^2 = 12, b^2 = 4$

$c^2 = 16$; $a^2m^2+b^2 = (12)(1) + 4 = 16$

Thus c2 = $a^2m^2+b^2$

$\therefore$ x-y+4=0 is a tangent to the ellipse.

The point of contact is $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$

i.e., $\left(\frac{-12}{4}, \frac{4}{4}\right)$ i.e., (-3, 1)

(7) Find the equation to the chord of contact from the point

(i) (-3, 1) to the parabola $y^2 = 8x$

(ii) (2, 4) to the ellipse $2x^2 + 5y^2 = 20$

(iii) (5, 3) to the hyperbola $4x^2 - 6y^2 = 24$

Solution :

(i) The equation of the chord of contact of tangents to the parabola

$y^2 = 8x$ from the point (x_1, y_1) is $yy_1 = 4(x+x_1)$

Here $(x_1, y_1) = (-3, 1)$

$\therefore y(1) = 4(x - 3)$

$4x - y - 12 = 0$ is the required equation.

(ii) The equation of the chord of contact of tangents to the ellipse from the point (x_1, y_1) is

$2xx_1 + 5yy_1 = 20$

Here (x_1, y_1) is (2, 4)

$\therefore 4x + 20y = 20$

$x + 5y - 5 = 0$ is the required equation.

(iii) Equation of the chord of contact of tangents to the hyperbola from the point (x_1, y_1) is $4xx_1 - 6yy_1 = 24$. Here (x_1, y_1) is (5, 3)

$20x - 18y = 24$

$10x - 9y\ 12 = 0$

EXERCISE 4.5

1. Find the equation of the asymptotes to the hyperbola

(i) $36x^2 - 25y^2 = 900$ (ii) $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

Solution:

(i) Equation of the hyperbola is $\frac{x^2}{25}$ - $\frac{y^2}{36}$ = 1

Combined equation of the asymptotes is $\frac{x^2}{25}$ - $\frac{y^2}{36}$ = 0

Separate equations of asymptotes are $\frac{x}{5}$ + $\frac{y}{6}$ = 0 and $\frac{x}{5}$ - $\frac{y}{6}$ = 0

(ii) $8x^2 + 10xy – 3y2 – 2x + 4y – 2 = 0$ is the equation of the hyperbola.

The combined equation of the asymptotes differs from the

hyperbola

by a constant only.

` The combined equation of the asymptotes is

$8x^2 + 10xy – 3y^2 – 2x + 4y + k = 0$

Consider $8x^2 + 10xy – 3y^2 = 8x^2 + 12xy – 2xy – 3y^2$

$= (4x – y) (2x + 3y)$

` The separate equations are of the form $4x – y + l = 0$; $2x + 3y + m = 0$

`$8x^2 + 10xy – 3y^2 – 2x + 4y + k = (4x – y + l) (2x + 3y + m)$

Equating the coefficients of x, y terms

$4m + 2l = -2$

$-m + 3l = 4$

Solving, we get $l = 1$ and $m = -1$

$lm = k = -1$

`the separate equations of asymptotes are $4x - y + 1 = 0$ and $2x + 3y - 1 = 0$

2. Find the equation of the hyperbola if

(i) the asymptotes are $2x + 3y - 8 = 0$ and $3x - 2y + 1 = 0$ (5, 3) is a point on the hyperbola

(ii) its asymptotes are parallel to $x + 2y - 12 = 0$ and $x - 2y + 8 = 0$, (2, 4) is the centre of the hyperbola and it passes through (2, 0).

Solution:

(i) The equation of the asymptotes are $2x + 3y - 8 = 0$ and $3x - 2y + 1 = 0$

` Equation of the hyperbola is of the form

$(2x + 3y - 8)(3x - 2y + 1) + k = 0$

But this passes through (5, 3)

` $(10 + 9 - 8)(15 - 6 + 1) + k = 0 \Rightarrow k = 110$

` The equation of the hyperbola is $(2x + 3y - 8)(3x - 2y + 1) = 110$

(ii) The asymptotes are parallel to $x + 2y - 12 = 0$ and $x - 2y + 8 = 0$

` Equations of the two asymptotes are of the form

$x + 2y + l = 0$ and $x - 2y + m = 0$

But asymptotes pass through the centre (2, 4) of the hyperbola.

$$2 + 8 + l = o \Rightarrow l = -10$$

$$2 - 8 + m = 0 \Rightarrow m = 6$$

` Equations of the asymptotes are $x + 2y - 10 = 0$ and $x - 2y + 6 = 0$

Combined equation of the asymptotes is $(x + 2y - 10)(x - y + 6) = 0$

` The equation of the hyperbola is of the form

$(x + 2y - 10)(x - 2y + 6) + k = 0$

It passes through (2, 0)

$(2 - 10)(2 + 6) + k = 0$

$$K = 64$$

Equation of the hyperbola is $(x + 2y - 10)(x - 2y + 6) + 64 = 0$

3. Find the angle between the asymptotes of the hyperbola

(i) 25x2 – 8y2 = 27

(ii) $4x^2 - 5y^2 - 16x + 10y + 31 = 0$

Solution:

(i) Equation of the hyperbola is $\frac{x^2}{27\,/\,24}\frac{y^2}{27\,/\,28} = 1$

Here $a^2 = \frac{27}{24} = \frac{9}{8} \Rightarrow a = \frac{3}{2\sqrt{2}}$

$b^2 = \frac{27}{8} \Rightarrow b = \frac{\sqrt[3]{3}}{\sqrt[2]{2}}; \frac{b}{a} = \sqrt{3}$

Angle between the asymptotes is $2\alpha = 2\tan^{-1}\frac{b}{a} = \tan^{-1}(\sqrt{3}) = 2\frac{\pi}{3} = \frac{2\pi}{3}$ $\left[\ \right]$

(ii) Equation of the hyperbola is $9(x-2)^2 - 4(y+3)^2 = 36$

$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

Here $a^2 = 4 \Rightarrow a = 2$

$b^2 = 9 \Rightarrow b = 3$

Angle between the asymptotes is $2\alpha = 2\tan^{-1}\frac{3}{2}$ $\left[\ \right]$

(iii) Equation of the hyperbola is $4x+ - 5y^2 - 16x + 10y + 31 = 0$

$4(x^2-4x) - 5(y^2-2y) = -31$

$4[(x-2)^2 - 4] - 5[(y-1)^2 - 1] = -31$

$4(x-2)^2 - 5(y-1)^2 = -20$

$5(y-1)^2 - 4(x-2)^2 = 20$

$\frac{(y-1)^2}{4} - \frac{(x-2)^2}{5} = 1$

$a^2 = \Rightarrow a = 2$

$b^2 = \Rightarrow b = \sqrt{5}$

EXERCISE 4.6

1. Find the equation of the rectangular hyperbola whose centre is

- $\left(\frac{1}{2}, \frac{1}{2}\right)$ and which passes through the point $\left(1, \frac{1}{4}\right)$

Solution:

The General form of standard rectangular hyperbola with centre at

(h, k) is $(x-h)(y-k) = c^2$

The centre is at $\frac{-1}{2}, \frac{-1}{2}$

` The equation of the R.H is $\left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = c^2$

But it passes through $\left(1+\frac{1}{4}\right)$

$\left(1+\frac{1}{2}\right)\left(\frac{1}{4}+\ \frac{1}{2}\right)$ = c^2

$\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)$ = c^2

$\Rightarrow$ $C^2 = \frac{9}{8}$

`The required equation of the R.H. is $\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = \frac{9}{8}$

2) Find the equation of the tangent and normal (i) at (3, 4) to the rectangular hyperbolas xy = 12 (ii) at $\left(-2,\frac{1}{4}\right)$ to the rectangular hyperbola 2xy-2x-8y-1=0

Solution :

(i) Equation of the tangent at (x_1, y_1) to a rectangular hyperbola is $xy_1+yx_1 =2c^2$

At (3, 4) the equation is 4x+3y=24

4x + 3y – 24 = 0 is the equation of the tangent.

Equation of the normal is of the form

3x -4yk=0

It passes through (3, 4) ` 3(3) -4(4)+k=0 k=7

` Equation of the normal is $3x - 4y + 7 = 0$

(ii) at $-2, \frac{1}{4}$ to the rectangular hyperbola $2xy - 2x - 8y - 1 = 0$

Note : (1) Equation of tangent and normal can be easily obtained by the calculus method (see section 5.1)

(2) This problem can also be done by replacing x by $\frac{x+x_1}{2}$ and y by $\frac{y+y_1}{2}$

Solution: $2xy - 2x - 8y - 1 = 0$

Differentiating with respect to x

$2(xy + y, 1)-2 -8y = 0$

$\Rightarrow y = \frac{2-2y}{2x-8}$

At $-2, \frac{1}{2}$, $y = -\frac{1}{8}$

Equation of the tangent is $(y - y_1) = m (x - x_1) \Rightarrow x + 8y = 0$

$\Rightarrow y - \frac{1}{4} = 8 (x + 2) \Rightarrow 32x - 4y + 65 = 0$

3. Find the equation of the rectangular hyperbola which has for one of

its asymptotes the line $x + 2y - 5 = 0$ and passes through the points (6, 0) and (-3, 0).

Solution: One of the asymptotes is $x + 2y - 5 = 0$

` The other asymptotes is of the form $2x - y + k = 0$

Equation of the R.H. is

$$(x + 2y - 5)\ (2x - y + k) + c = 0$$

It passes through (6, 0)

$$(6 - 5)\ (12 + k) + c = 0$$

$$k + c = -12 \qquad .....(1)$$

It also passes through (-3, 0)

$$(-3 - 5)\ (-6 + k) + c = 0$$

$$(-8)\ (-6 + k) + c = 0$$

$$48 - 8k + c = 0$$

$$-8k + c = -48 \qquad .....(2)$$

Solving (1) and (2) $k = 4$ and $c = -16$

` Equation of the R.H. is $(x + 2y - 5)\ (2x - y + 4) - 16 = 0$

4. A standard rectangular hyperbola has its vertices at (5,7) and (-3, -1).

Find its equation and asymptotes.

Solution:

Vertices of the rectangular hyperbola are A (5, 7) and A' (-3, -1)

$$AA' = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}$$

$$2a = 8\sqrt{2} \Rightarrow a = 4\sqrt{2}$$

$$c^2 = \frac{a^2}{2} = c^2 = \frac{32}{2} = 16$$

The centre is the midpoint of AA' = (1,3)

Equation of the R.H. with centre 91,3) is

$$(x-1)(y-3) = 16$$

The combined equation of the asymptotes is

$$(x-1)(y-3) = 0$$

Separate equations are $x - 1 = 0$ and $y - 3 = 0$

5. Find the equation of the rectangular hyperbola which has its centre at (2, 1), one of its asymptotes $3x - y - 5 = 0$ and which passes through the point (1, -1)

Solution:

One of the asymptotes is $3x - y - 5 = 0$

The other asymptote is of the form $x + 3y + k = 0$

` Combined equation of the asymptotes is $(3x - y - 5)(x + 3y - 5) = 0$

Equations of the rectuangular hyperbola is of the form

$(3x - y - 5)(x + 3y - 5) + k = 0$

But it passes through (1, -1)

$$\therefore (3+1-5)(x+3y-5)\, k = 0 \implies k = -7$$

$\therefore$ Equation of the required rectangular hyperbola is

$$(3x-y-5)(x+3y-5) - 7 = 0$$

(6) Find the equations of the asymptotes of the following rectangular hyperbolas.

(i) $xy - kx - hy = 0$ (ii) $2xy + 3x + 4y + 1 = 0$

(iii) $6x^2 + 5xy - 6y^2 + 12 + 5y + 3 = 0$

Solution :

(i) Equation of the rectangular hyperbola is $xy - kx - hy = 0$

Equation of asymptotes is $xy - kx - hy + A = 0$

Consider

$$(x-1)(y-m) = xy - kx - hy + A$$

$$\implies 1 = h,\ m = k,\ A = lm = hk$$

$\therefore$ Equation of the asymptotes is

$$xy-kx-hy+hk=0$$

i.e., $(x-h)(y-k)=0$

The separate equations are $x-h = 0$ and $y-k=0$

(ii) Equation of the asymptotes to the given RH is

$$2xy+3x+4y+k=0$$

$(2x+1)(y+m) = 2xy+3x+4y+k$

$2m = 3$; $1=4$, $1m = k$

i.e., $m = \frac{3}{2}$, $l = 4$, $k = 6$

Combined equation is

$$2xy+3x+4y+6=0$$

i.e., $(2x+4)\left(y+\frac{3}{2}\right) = 0$

Separate equations are $x + 2 = 0$; $y + \frac{3}{2} = 0$

(iii) The combined equation of the asymptotes of the form

$$6x^2 + 5xy - 6y^2 + 12x + 5y + k = 0$$

Consider $6x^2 + 5xy - 6y^2 = (3x - 2y)(2x + 3y)$

$6x^2 + 5xy - 6y^2 + 12x + 5y + k = (3x - 2y + l)(2x + 3y + m)$

Comparing the terms of x and y

$12 = 3m + 21$

$5 = -2m + 3l$

Solving them, $l = 3$ and $m = 2$; $k = lm = 6$

Separate equations are

$$3x - 2y + 3 = 0$$

$$2x + 3y + 2 = 0$$

(7) Prove that the tangent at any point to the rectangular hyperbola forms with the asymptotes a triangle of constant area.

Solution :

Equation of the tangent at 't' on the R.H is

$$x + yt^2 = 2ct$$

$$\text{i.e., } \frac{x}{2ct} + \frac{y}{(2clt)} = 1$$

$\therefore$ The x and y intercepts are 2ct and $\frac{2c}{t}$ respectively.

Assume that the tangent meets x and y axes at P and Q respectively.

$\therefore$ The coordinates of P and Q are (2ct, 0) and $\left(0, \frac{2c}{t}\right)$

$\therefore$ The required area of the ΔOPQ

$$= \frac{1}{2}\, x\ OP\ x\ OP$$

$$= \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 2c^2 \text{ (a constant)}$$